5.2 HW: Worked Solutions

p250–251: 13, 15, 19, 25, 31

13. Find the general solution where differentiation is with respect to t.

$$\frac{dx}{dt} = x - 4y$$

$$\frac{dy}{dt} = x + y$$

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 1 & -4 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{bmatrix} 1-r & -4 \\ 1 & 1-r \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(1-r)^2 + 4 = 0$$

$$1 - 2r + r^2 + 4 = 0 \qquad r^2 - 2r + 5 = 0$$

$$r = \frac{2 \pm \sqrt{4-20}}{2} = 1 \pm \frac{4}{2}i = \underset{\alpha}{\underset{\shortparallel}{1}} \pm \underset{\beta}{\underset{\shortparallel}{2}}i$$

$r_1 = 1 + 2i$

$$\begin{pmatrix} 1-(1+2i) & -4 \\ 1 & 1-(1+2i) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -2i & -4 \\ 1 & -2i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-2ix - 4y = 0$$

$$y = -\tfrac{1}{2}ix \qquad \vec{u}_1 = \begin{bmatrix} -2 \\ i \end{bmatrix} = \overset{a}{\overset{\shortparallel}{\begin{bmatrix} -2 \\ 0 \end{bmatrix}}} + \overset{b}{\overset{\shortparallel}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}}\, i$$

$$\begin{pmatrix} X \\ y \end{pmatrix} = c_1\left[e^t\cos 2t\begin{bmatrix} -2 \\ 0 \end{bmatrix} - e^t\sin 2t\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right]$$

$$+ c_2\left[e^t\sin 2t\begin{bmatrix} -2 \\ 0 \end{bmatrix} + e^t\cos 2t\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right]$$

$$X(t) = -2c_1e^t\cos 2t - 2c_2e^t\sin 2t$$

$$y(t) = -c_1e^t\sin 2t + c_2e^t\cos 2t$$

15. $$\frac{dw}{dt} = 5w + 2z + 5t$$

$$\frac{dz}{dt} = 3w + 4z + 17t$$

$$\begin{pmatrix} w \\ z \end{pmatrix}' = \begin{pmatrix} 5 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} w \\ z \end{pmatrix} + \begin{pmatrix} 5t \\ 17t \end{pmatrix}$$

$$\begin{vmatrix} 5-r & 2 \\ 3 & 4-r \end{vmatrix} = (5-r)(4-r) - 6 = 0$$

$$20 - 9r + r^2 - 6 = 0$$

$$r^2 - 9r + 14 = 0$$

$$(r-7)(r-2) = 0$$

$r_1 = 7 \quad r_2 = 2$

r_1:
$$\begin{bmatrix} -2 & 2 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} w \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$w = z \qquad \vec{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}s$$

r_2:
$$\begin{bmatrix} 3 & 2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} w \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$3w = -2z \qquad \vec{u}_2 = \begin{bmatrix} 2 \\ -3 \end{bmatrix}s$$

$\begin{pmatrix} w \\ z \end{pmatrix}_p = at + b$

$\begin{pmatrix} w \\ z \end{pmatrix}_p' = a$

$a = \begin{pmatrix} 5 & 2 \\ 3 & 4 \end{pmatrix}(at + b) + \begin{pmatrix} 5t \\ 17t \end{pmatrix}$

$\boxed{a = Aat + Ab + \begin{pmatrix} 5t \\ 17t \end{pmatrix}}$

$Aa = -\begin{pmatrix} 5 \\ 17 \end{pmatrix}$ $\quad \begin{pmatrix} 5 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 5a_1 + 2a_2 \\ 3a_1 + 4a_2 \end{pmatrix} = \begin{pmatrix} -5 \\ -17 \end{pmatrix}$

$a = Ab$

$5a_1 + 2a_2 = -5$

$3a_1 + 4a_2 = -17$

$-7a_1 = -7 \quad a_1 = 1$

$5(1) + 2a_2 = -5$

$a_2 = -5$

$\begin{bmatrix} 1 \\ -5 \end{bmatrix} = \begin{pmatrix} 5b_1 + 2b_2 \\ 3b_1 + 4b_2 \end{pmatrix}$

$5b_1 + 2b_2 = 1$

$3b_1 + 4b_2 = -5$

$-7b_1 = -7 \quad b_1 = 1$

$b_2 = -2$

General $\begin{pmatrix} w \\ z \end{pmatrix} = c_1 e^{7t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{2t}\begin{bmatrix} 2 \\ -3 \end{bmatrix} + \begin{bmatrix} 1 \\ -5 \end{bmatrix}t + \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$$\boxed{\begin{aligned} w(t) &= c_1 e^{7t} + 2c_2 e^{2t} + t + 1 \\ z(t) &= c_1 e^{7t} - 3c_2 e^{2t} - 5t - 2 \end{aligned}}$$

Solve the IVP

19. $\frac{dx}{dt} = 4x + y \quad x(0) = 1$

$\frac{dy}{dt} = -2x + y \quad y(0) = 0$

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 4 & 1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{vmatrix} 4-r & 1 \\ -2 & 1-r \end{vmatrix} = (4-r)(1-r) + 2 = 0$$

$$4 - 5r + r^2 + 2 = 0$$

$$r^2 - 5r + 6 = 0$$

$$(r-3)(r-2) = 0$$

$r_1 = 3 \quad r_2 = 2$

r_1: $\begin{pmatrix} 1 & 1 \\ -2 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$x = -y \qquad \vec{u_1} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} s$

r_2: $\begin{pmatrix} 2 & 1 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$2x = -y \qquad \vec{u_2} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} s$

$y = -2x$

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{2t}$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$-c_1 + c_2 = 1 \qquad -c_2 = 1 \quad c_2 = -1$

$c_1 - 2c_2 = 0 \quad c_1 = 2c_2 \qquad c_1 = -2$

$$\begin{pmatrix} x \\ y \end{pmatrix} = -2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{3t} - \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{2t}$$

$$\boxed{\begin{aligned} x(t) &= 2e^{3t} - e^{2t} \\ y(t) &= -2e^{3t} + 2e^{2t} \end{aligned}}$$

25. $x' = x + 2y - z$

$y' = x + z$

$z' = 4x - 4y + 5z$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}' = \begin{pmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{vmatrix} 1-r & 2 & -1 \\ 1 & -r & 1 \\ 4 & -4 & 5-r \end{vmatrix} = (1-r)\begin{vmatrix} -r & 1 \\ -4 & 5-r \end{vmatrix} - 2\begin{vmatrix} 1 & 1 \\ 4 & 5-r \end{vmatrix} - \begin{vmatrix} 1 & -r \\ 4 & -4 \end{vmatrix}$$

$$= (1-r)\left[-5r + r^2 + 4\right] - 2\left[\underset{1-r}{5 - r - 4}\right] - \left[-4 + 4r\right]$$

$$= -5r + r^2 + 4 + 5r^2 - r^3 - 4r - 2 + 2r + 4 - 4r = 0$$

$$-r^3 + 6r^2 - 11r + 6 = 0$$

$$r^3 - 6r^2 + 11r - 6 = 0$$

$r = 1$

$(r-1)(r^2 + ar + b) = 0$

$\cancel{r^3} + ar^2 + br - r^2 - ar - b = \quad \cancel{r^3} - 6r^2 + 11r - 6$

$b = 6$

$a - 1 = -6$

$a = -5$

$(r-1)(r^2 - 5r + 6)$

$= (r-1)(r-3)(r-2)$

$r_1 = 1,\ r_2 = 3,\ r_3 = 2$

r_1:

$$\begin{pmatrix} 0 & 2 & -1 \\ 1 & -1 & 1 \\ 4 & -4 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$2y = z \qquad \vec{u_1} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} s$$

$$x - y + z = 0$$

$$x - y + 2y = 0$$

$$x = -y$$

r_2:

$$\begin{pmatrix} -2 & 2 & -1 \\ 1 & -3 & 1 \\ 4 & -4 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -4 & 1 \\ 1 & -3 & 1 \\ 4 & -4 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$z = 4y$$

$$x - 3y + z = 0 \qquad \vec{u_2} \begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix} s$$

$$x - 3y + 4y = 0$$

$$x = -y$$

r_3:

$$\begin{pmatrix} -1 & 2 & -1 \\ 1 & -2 & 1 \\ 4 & -4 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 2 & -1 \\ 1 & -2 & 1 \\ 0 & 4 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-x + 2y - z = 0$$

$$4y = z \qquad \vec{u_3} = \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} s$$

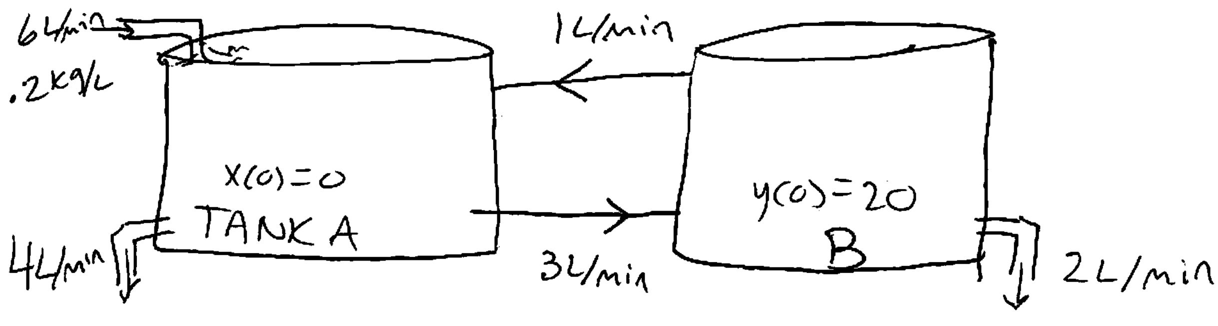
$$-x + 2y - 4y = 0$$

$$x = -2y$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = c_1 e^{t} \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix} + c_3 e^{2t} \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}$$

$$x = -c_1 e^{t} - c_2 e^{3t} - 2c_3 e^{2t}$$
$$y = c_1 e^{t} + c_2 e^{3t} + c_3 e^{2t}$$
$$z = 2c_1 e^{t} + 4c_2 e^{3t} + 4c_3 e^{2t}$$

31. 2 large tanks each hold 100L of liquid are interconnected by pipes with the liquid flowing from tank A into tank B at a rate of 3 L/min & from B into A at a rate of 1L/min. The liquid is kept well-stirred. A brine solution with a concentration of .2kg/L of salt flows into tank A at a rate of 6 L/min. The solution flows out of the system from tank A at 4L/min & from tank B at 2L/min. If, initially, tank A contains pure water & tank B contains 20 kg of salt, determine the mass of salt in each tank at time t.



* Notice the volume in the tanks is not changing

$X' = \text{Rate In} - \text{Rate out}$

$y' = \text{Rate In} - \text{Rate out}$

$$X' = (.2\,kg/L)(6\tfrac{L}{min}) + \left(\frac{y}{100}\right)(1L/min) - \frac{X}{100}\left(\frac{4L}{min}\right) - \frac{X}{100}\left(\frac{3L}{min}\right)$$

$$X' = 1.2 + \frac{y}{100} - \frac{7X}{100}$$

$$y' = \frac{X}{100}(3L/min) - \frac{y}{100}\left(\frac{1L}{min}\right) - \frac{y}{100}\left(\frac{2L}{min}\right)$$

$$= \frac{3X}{100} - \frac{3y}{100}$$

$$\begin{pmatrix} X \\ y \end{pmatrix}' = \begin{pmatrix} \frac{-7}{100} & \frac{1}{100} \\ \frac{3}{100} & \frac{-3}{100} \end{pmatrix} \begin{pmatrix} X \\ y \end{pmatrix} + \begin{pmatrix} 1.2 \\ 0 \end{pmatrix}$$

$$\begin{vmatrix} -\frac{7}{100} - r & \frac{1}{100} \\ \frac{3}{100} & \frac{-3}{100} - r \end{vmatrix} = \left(\frac{-7}{100} - r\right)\left(\frac{-3}{100} - r\right) - \frac{3}{100^2}$$

$$\frac{21}{100^2} + \frac{10r}{100} + r^2 - \frac{3}{100^2}$$

$$r^2 + \frac{10r}{100} + \frac{18}{100^2}$$

$$r = \frac{\frac{-10}{100} \pm \sqrt{\frac{100}{100^2} - \frac{72}{100^2}}}{2} = \frac{\frac{-10}{100} \pm \frac{\sqrt{28}}{100}}{2} =$$ sadly these are the correct values of r.

For a test:

$$\boxed{\begin{array}{l} \begin{pmatrix} X \\ y \end{pmatrix}' = \begin{pmatrix} \frac{-7}{100} & \frac{1}{100} \\ \frac{3}{100} & \frac{-3}{100} \end{pmatrix} \begin{pmatrix} X \\ y \end{pmatrix} + \begin{pmatrix} 1.2 \\ 0 \end{pmatrix} \\ \begin{pmatrix} X(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 20 \end{pmatrix} \end{array}}$$