7.7/7.8 2nd order linear differential equations

To solve: $ay'' + by' + cy = 0$

1st write auxillary (characteristic) equation:

$$ar^2 + br + c = 0$$

$$r = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and } \frac{-b-\sqrt{b^2-4ac}}{2a}$$

* If $b^2-4ac > 0$, the general solution is $y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$
* If $b^2-4ac = 0$, we will get only one value for r and the general solution is $y = c_1 e^{rx} + c_2 x e^{rx}$
* If $b^2-4ac < 0$, then the general solution is $y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$ where

$$\alpha = \frac{-b}{2a} \quad \beta = \frac{\sqrt{4ac-b^2}}{2a}$$

Ex 1 Find the general solution to

$y''-4y'+4y=0$

$r^2-4r+4=0$

$(r-2)^2=0$

$r=2$ repeated roots

$$\boxed{y=C_1e^{2x}+C_2xe^{2x}}$$

Ex 2 Solve the Initial Value Problem

$y''+4y'+3y=0 \quad y(0)=0 \quad y'(0)=9$

$r^2+4r+3=0$

$(r+3)(r+1)=0$

$y=C_1e^{-3x}+C_2e^{-x}$ (If this problem asked for the general solution, we'd stop here.)

$y(0)=C_1e^{-3(0)}+C_2e^{-(0)}=C_1e^0+C_2e^0=C_1+C_2=0$

$y'=-3C_1e^{-3x}-C_2e^{-x}$

$y'(0)=-3C_1-C_2=9$

$C_1=-C_2$

$-3C_1-C_2=9$

$-3(-C_2)-C_2=9 \rightarrow 2C_2=9 \quad C_2=9/2 \quad C_1=-9/2$

$$\boxed{y=-\frac{9}{2}e^{-3x}+\frac{9}{2}e^{-x}}$$

To solve: $ay'' + by' + cy = f(x)$

Use the Method of Undetermined Coefficients

1. Find y_c the general solution to $ay'' + by' + cy = 0$
2. Find the form y_p of a particular solution to $ay'' + by' + cy = f(x)$
3. Check to see if y_p overlaps with y_c, if it does multiply y_p by x. If it still overlaps multiply y_p by another x.
4. Find y''_p and y'_p and plug that along with y_p into $ay'' + by' + cy = f(x)$. Solve for A, B, etc.
5. The general solution to $ay'' + by' + cy = f(x)$ is $y = y_c + y_p$
6. If they give initial conditions use them along with $y = y_c + y_p$ to find c_1 & c_2.

Ex 3 Use the differential equation $y'' + 9y = f(x)$ along with the value of $f(x)$ listed below to find the form of the particular solution. Do NOT solve for coefficients.

a) $x^3 - 6x$

b) $4\sin 2x$

c) e^{-x}

d) $4\cos 3x$

e) $e^{-x}\sin 3x$

* First we need y_c

$r^2 + 9 = 0 \rightarrow r = \pm\sqrt{-9} = \pm 3i \qquad \alpha = 0,\ \beta = 3$

$y_c = e^{0x}(c_1\cos 3x + c_2\sin 3x) = c_1\cos 3x + c_2\sin 3x$

a) * We match the degree of the polynomial & include every term

$\boxed{y_p = Ax^3 + Bx^2 + Cx + D}$

b) * If we have $\sin ax$ or $\cos ax$ we try $y_p = A\cos ax + B\sin ax$

So for this ex. $y_p = A\cos 2x + B\sin 2x$

We check to see if it overlaps y_c and it doesn't. $\boxed{y_p = A\cos 2x + B\sin 2x}$

c) * Try $y_p = Ae^{-x}$ & check for overlap in y_c. There is no overlap

$$\boxed{y_p = Ae^{-x}}$$

d) $y_p = A\cos 3x + B\sin 3x$ but $y_c = C_1\cos 3x + C_2\sin 3x$

y_p overlaps y_c. If we plug y_p & its derivatives into $y'' + 9y = 4\cos 3x$ we'll get a contradiction,

Try multiplying y_p by x

$$\boxed{y_p = Ax\cos 3x + Bx\sin 3x}$$

Now we no longer are overlapping

e) * Think about what you would try for e^{-x} and $\sin 3x$, basically we just combine them.

$$\boxed{y_p = e^{-x}(A\cos 3x + B\sin 3x)}$$

This doesn't overlap y_c

(Note: $y_p = Ce^{-x}(A\cos 3x + B\sin 3x)$
$= e^{-x}((CA)\cos 3x + (CB)\sin 3x)$
The C isn't necessary.)

Ex ④ Find the general solution

$y'' + 2y' + y = x$

Find y_c:

$r^2 + 2r + 1 = 0$

$(r+1)^2 = 0$

$y_c = C_1 e^{-x} + C_2 x e^{-x}$

Find form of y_p:

$y_p = Ax + B$

No overlap!

$y'_p = A$

$y''_p = 0$

* Now plug into $y'' + 2y' + y = x$. For y_p to be a solution to this, when we plug y_p & its derivatives into the left we should get x *

$0 + 2A + (Ax + B) = x$

$2A + Ax + B = x$

$A = 1$

$2A + B = 0 \rightarrow B = -2$

$y_p = x - 2$

$$\boxed{y = y_c + y_p = C_1 e^{-x} + C_2 x e^{-x} + x - 2}$$

* If they gave initial conditions we could have found C_1 & C_2

EX(5) Find the general solution

$y'' + 2y' + y = 3e^{-x}$

* We found $y_c = C_1e^{-x} + C_2xe^{-x}$ in the previous ex

Try $y_p = Ae^{-x}$, but this overlaps C_1e^{-x}

$y_p = Axe^{-x}$, we've multiplied by an x but this still overlaps

$y_p = Ax^2e^{-x}$ No longer overlaps

$y'_p = 2Axe^{-x} - Ax^2e^{-x} = (2Ax - Ax^2)e^{-x}$

$y''_p = (2A - 2Ax)e^{-x} - (2Ax - Ax^2)e^{-x}$

$= (2A - 2Ax - 2Ax + Ax^2)e^{-x}$

$= (2A - 4Ax + Ax^2)e^{-x}$

Plug in to $y'' + 2y' + y = 3e^{-x}$

$e^{-x}[2A - 4Ax + Ax^2 + 2(2Ax - Ax^2) + Ax^2] = 3e^{-x}$

$e^{-x}[2A] = 3e^{-x}$ $\quad 2A = 3 \quad A = 3/2$

$y_p = \frac{3}{2}x^2e^{-x}$

$$\boxed{y = y_c + y_p = C_1e^{-x} + C_2xe^{-x} + \frac{3}{2}x^2e^{-x}}$$

Ex 6 Find the form of y_p but do <u>NOT</u> solve for coefficients if

$$y'' + 2y' + y = e^{4x} - \cos 3x + x^2 - 4$$

$* \; y_c = c_1 e^{-x} + c_2 x e^{-x}$

We use superposition principle to find the form for y_p

$$y_p = Ae^{4x} + B\cos 3x + C\sin 3x + Dx^2 + Ex + F$$

There is no overlap, so this is our answer.

Ex 7 Find the form of y_p but do NOT solve for coefficients if

$$y'' + 2y' + y = 3e^{-x} + \sin 2x + 5$$

$* \; y_c = c_1 e^{-x} + c_2 x e^{-x}$

Using superposition again

$$y_p = Ae^{-x} + B\cos 2x + C\sin 2x + D$$

But this time we have overlap on the first term Ae^{-x}

so

$$y_p = Ax^2 e^{-x} + B\cos 2x + C\sin 2x + D$$

* Try the IVPs suggested on the Review Sheet and look over the ones we've worked in class