9.5 Homogeneous Linear Systems with Constant Coefficients

* We are trying to solve $\vec{x}' = A\vec{x}$ *

Technique: Solving $\vec{x}' = A\vec{x}$ when we have real eigenvalues

① Find all eigenvalues by setting $|A - rI| = 0$ and solving for r

② For each eigenvalue r_i find its associated eigenvector $\vec{u}_i$ which satisfies $(A - rI)\vec{u} = \vec{0}$

③ The general solution to $\vec{x}' = A\vec{x}$ is $\vec{x} = c_1 e^{r_1 t}\vec{u}_1 + c_2 e^{r_2 t}\vec{u}_2 + \cdots + c_n e^{r_n t}\vec{u}_n$ where $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n$ are your eigenvectors and $r_1, \ldots, r_n$ the eigenvalues

Examples from Differential Equations by Polking, Boggess, Arnold p463-465; 490

For the given matrix A and initial condition $\vec{x}(0)$ solve the IVP

$\vec{x}' = A\vec{x}$ $\quad \vec{x}(0) = [\ ]$

① $A=\begin{pmatrix}2 & -6\\ 0 & -1\end{pmatrix}$ $\quad\vec{x}(0)=\begin{bmatrix}0\\1\end{bmatrix}$

Step 1: Find eigenvalues r_i

$$\begin{vmatrix}2-r & -6\\ 0 & -1-r\end{vmatrix}=0$$

$$\begin{vmatrix}2-r & -6\\ 0 & -1-r\end{vmatrix}=(2-r)(-1-r)-0(-6)=0$$

$$(2-r)(-1-r)=0$$

Eigenvalues: $r_1=2,\ r_2=-1$

Step 2: Find eigenvectors $\vec{u}_i$

First we'll find $\vec{u}_1$

$$(A-2I)\vec{u}_1=\vec{0}$$

$$\begin{pmatrix}0 & -6\\ 0 & -3\end{pmatrix}\begin{pmatrix}u_1\\u_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

*This gives us $-6u_2=0$ and $-3u_2=0$
Both give the same information. This tells us $u_2=0$. We have no requirements on u_1, so $u_1=1s$ where s is some arbitrary value.*

So $\vec{u}_1=\begin{pmatrix}1\\0\end{pmatrix}s$ $\quad u_1 \quad u_2$

Now we'll find $\vec{u}_2$

$$(A-(-1)I)\vec{u}_2=\vec{0}$$

$$\begin{pmatrix}3 & -6\\ 0 & 0\end{pmatrix}\begin{pmatrix}u_1\\u_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

P3

If we multiply we get $3u_1 - 6u_2 = 0$
or $u_1 = 2u_2$

* Let $u_2 = 1$ (we could have picked any nonzero number but it is smart to pick one which will make your life better)

Since $u_1 = 2u_2$ if $u_2 = 1$ then
$u_1 = 2$

so $\vec{u}_2 = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} s$

Our Eigenvectors are $\vec{u}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} s$, $\vec{u}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} s$

$\boxed{\text{The general solution is } \vec{x} = c_1 e^{2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}}$

* Notice we drop the s here
c_1 & c_2 can pick up where s left off *

Since we were asked to solve the IVP we have to go further
$\vec{x}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ so plug 0 in for t in the general solution

$\vec{x}(0) = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

or $c_1 + 2c_2 = 0$

$0 + c_2 = 1 \rightarrow c_1 = -2$

$$\boxed{\text{Our solution is } \vec{x} = -2e^{2t}\begin{pmatrix}1\\0\end{pmatrix} + e^{-t}\begin{pmatrix}2\\1\end{pmatrix}}$$

Ex 2 $\quad A = \begin{pmatrix}-2 & 0\\ 0 & -2\end{pmatrix} \quad \vec{x}(0) = \begin{pmatrix}3\\-2\end{pmatrix}$

$$\begin{vmatrix}-2-r & 0\\ 0 & -2-r\end{vmatrix} = (-2-r)(-2-r) - 0 = 0$$

$$(2+r)^2 = 0$$

$r=-2$ is the only eigenvalue, it has multiplicity 2

* Because A is symmetric we are guaranteed 2 l.i. eigenvectors. If it wasn't symmetric, it is possible we would have only found 1 eigenvector -- unfortunately we won't be dealing with that case *

$$(A-(-2)I)\vec{u} = \vec{0}$$

$$\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}u_1\\u_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

We have no requirements on u_1 or u_2 since any values of u_1 or u_2 will

P5

satisfy $0u_1 + 0u_2 = 0$

Our eigenvectors are $\vec{u_1} = \begin{pmatrix}1\\0\end{pmatrix}$s ← let's u_1 be anything

and $\vec{u_2} = \begin{pmatrix}0\\1\end{pmatrix}$s

let's u_2 be anything

The general solution is:

$$X = c_1 e^{-2t}\begin{pmatrix}1\\0\end{pmatrix} + c_2 e^{-2t}\begin{pmatrix}0\\1\end{pmatrix}$$

To solve the IVP we need to satisfy our given initial condition $\vec{X}(0) = \begin{pmatrix}3\\-2\end{pmatrix}$

$$\begin{pmatrix}3\\-2\end{pmatrix} = c_1 e^{-2(0)}\begin{pmatrix}1\\0\end{pmatrix} + c_2 e^{-2(0)}\begin{pmatrix}0\\1\end{pmatrix}$$

$$= c_1\begin{pmatrix}1\\0\end{pmatrix} + c_2\begin{pmatrix}0\\1\end{pmatrix}$$

$c_1 = 3 \quad c_2 = -2$

$$\boxed{X = 3e^{-2t}\begin{pmatrix}1\\0\end{pmatrix} - 2e^{-2t}\begin{pmatrix}0\\1\end{pmatrix}}$$

Ex3 Find the general solution $\vec{x}' = A\vec{x}$

$$A = \begin{pmatrix}2 & 0 & 0\\-6 & 1 & -4\\-3 & 0 & -1\end{pmatrix}$$

$$|A - rI| = 0$$

P6

$$\begin{vmatrix} 2-r & 0 & 0 \\ -6 & 1-r & -4 \\ -3 & 0 & -1-r \end{vmatrix} = (2-r)\begin{vmatrix} 1-r & -4 \\ 0 & -1-r \end{vmatrix} - 0\begin{vmatrix} -6 & -4 \\ -3 & -1-r \end{vmatrix}$$

$$+ 0\begin{vmatrix} -6 & 1-r \\ -3 & 0 \end{vmatrix}$$

$$= (2-r)\left[(1-r)(-1-r) - (-4)(0)\right] = 0$$

$$= (2-r)(1-r)(-1-r)$$

$$\boxed{r_1 = 2 \quad r_2 = 1 \quad r_3 = -1}$$ eigenvalues

$\vec{u}_1$: $\left(A - \overset{r=2}{2}I\right)\vec{u}_1 = \vec{0}$

$$\begin{pmatrix} 0 & 0 & 0 \\ -6 & -1 & -4 \\ -3 & 0 & -3 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\xrightarrow{r_2 - 2r_3} \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 2 \\ -3 & 0 & -3 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$-u_2 + 2u_3 = 0 \qquad u_2 = 2u_3$

$-3u_1 - 3u_3 = 0 \qquad u_1 = -u_3 \qquad$ let $u_3 = 1$ $\qquad u_2 = 2$, $u_1 = -1$

$$\vec{u}_1 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} s$$

P7

$r_2 = 1$

$\vec{u}_2$: $(A-I)\vec{u}_2 = \vec{0}$

$$\begin{array}{l} \\ \text{Repeat} \to \\ \text{info} \to \end{array} \begin{pmatrix} 1 & 0 & 0 \\ -6 & 0 & -4 \\ -3 & 0 & -2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

* This tells us $u_1 = 0$ & $-6u_1 - 4u_3 = 0$
so $u_3 = 0$

u_2 is unrest

$\vec{u}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} s$

$r_3 = -1$

$\vec{u}_3$: $(A-(-1)I)\vec{u}_3 = \vec{0}$

$$\begin{pmatrix} 3 & 0 & 0 \\ -6 & 2 & -4 \\ -3 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$3u_1 = 0 \to u_1 = 0$

$\cancel{-6u_1} + 2u_2 - 4u_3 = 0$

$u_2 = 2u_3$

let $u_3 = 1 \to u_2 = 2$

$\vec{u}_3 = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} s$

The general solution: $\boxed{X = C_1 e^{2t}\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} + C_2 e^{t}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + C_3 e^{-t}\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}}$

P9 Find the general solution

$$\vec{X}' = \begin{pmatrix} 2 & 0 & 0 \\ -6 & 2 & 3 \\ 6 & 0 & -1 \end{pmatrix} \vec{X}$$

$$\begin{vmatrix} 2-r & 0 & 0 \\ -6 & 2-r & 3 \\ 6 & 0 & -1-r \end{vmatrix} = (2-r)^2(-1-r) = 0$$

$r=2 \quad r=-1$

$\vec{u}_1$ & $\vec{u}_2$:

$$(A-2I)\vec{u} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ -6 & 0 & 3 \\ 6 & 0 & -3 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$-6u_1 + 3u_3 = 0$

$u_3 = 2u_1$ let $u_1 = 1 \rightarrow u_3 = 2$

u_2 is unrestricted

$$\vec{u}_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} s \qquad \vec{u}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} s$$

$\vec{u}_3$: $(A-(-1)I)\vec{u} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 3 & 0 & 0 \\ -6 & 3 & 3 \\ 6 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

P10

$u_1 = 0$

$3u_2 + 3u_3 = 0$

$u_2 = -u_3$

$\vec{u}_3 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} s$

Eigenvectors: $\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} s$, $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} s$, $\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} s$

General solution:

$$\boxed{X = c_1 e^{2t}\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + c_2 e^{2t}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3 e^{-t}\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}}$$