9.6 Complex Eigenvalues

Recall: When we are trying to find $\vec{x}$ that solves $\vec{x}' = A\vec{x}$, first we find our eigenvalues by using $|A - rI| = 0$ and solving for r. Once we have our eigenvalues, we find our eigenvectors by solving $(A - rI)\vec{u} = \vec{0}$ and solving for $\vec{u}$.

It is possible to find $r = \alpha \pm \beta i$ when solving $|A - rI| = 0$ for r. Complex numbers come in pairs, so if you find $r_1 = \alpha + \beta i$ there will also be an $r_2 = \alpha - \beta i$. Likewise when we find an eigenvector $\vec{u}_1 = \vec{a} + \vec{b}i$ associated with r_1 there will be an eigenvector $\vec{u}_2 = \vec{a} - \vec{b}i$ associated with r_2.

Using Euler's formula we found in class that two linearly independent real vector solutions to $\vec{x}' = A\vec{x}$ are

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$$\boxed{\begin{gathered} e^{\alpha t}\cos(\beta t)\vec{a} - e^{\alpha t}\sin(\beta t)\vec{b} \\ \text{and} \\ e^{\alpha t}\cos(\beta t)\vec{b} + e^{\alpha t}\sin(\beta t)\vec{a} \end{gathered}}$$

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So for $A_{2\times 2}$ the general solution to $\vec{x}' = A\vec{x}$ is

$$\vec{x} = c_1\left(e^{\alpha t}\cos(\beta t)\vec{a} - e^{\alpha t}\sin(\beta t)\vec{b}\right) + c_2\left(e^{\alpha t}\cos(\beta t)\vec{b} + e^{\alpha t}\sin(\beta t)\vec{a}\right)$$

Examples from Elementary Differential Equations and Boundary Value Problems by Boyce & DiPrima

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EX 1 Find the solution of the given IVP

$$\vec{x}\,' = \begin{pmatrix} 1 & -5 \\ 1 & -3 \end{pmatrix}\vec{x} \qquad \vec{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

* First we find the eigenvalues of $\begin{pmatrix} 1 & -5 \\ 1 & -3 \end{pmatrix}$

$$\left|\begin{pmatrix} 1 & -5 \\ 1 & -3 \end{pmatrix} - rI\right| = \begin{vmatrix} 1-r & -5 \\ 1 & -3-r \end{vmatrix} = (1-r)(-3-r)-(-5) = 0$$

$$= -3-r+3r+r^2+5 = 0$$

$$= r^2+2r+2 = 0$$

* Doesn't factor, use the quadratic formula

$$r = \frac{-2 \pm \sqrt{4-8}}{2} = \frac{-2 \pm \sqrt{-4}}{2} = \underbrace{-1}_{\alpha=-1} \pm \underbrace{i}_{\beta=1}$$

* Since our eigenvalues are complex we only need to use one of them to get all of our eigenvectors

$r = -1+i$ (I could have used the other)

$$(A - rI)\vec{u} = \vec{0}$$

$$(A-(-1+i)I)\vec{u} = \vec{0}$$

$$(A+(1-i)I)\vec{u} = 0$$

$$\begin{pmatrix} 2-i & -5 \\ 1 & -2-i \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

* Both rows of this 2×2 give the same information, use the simpler one

$u_1 + (-2-i)u_2 = 0$

$u_1 = (2+i)u_2$ * This describes how the components of our vector relate to each other

Let $u_2 = 1$ (any nonzero number would work, also we could have fixed u_1 & found u_2)

If $u_2 = 1$ then $u_1 = 2+i$ so

$\vec{u} = \begin{pmatrix} 2+i \\ 1 \end{pmatrix}$ * we need to write $\vec{u}$ as $\vec{a} + \vec{b}i = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} i = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} b_1 i \\ b_2 i \end{pmatrix}$

$= \begin{pmatrix} a_1 + b_1 i \\ a_2 + b_2 i \end{pmatrix}$

so $\vec{u} = \begin{pmatrix} 2+i \\ 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 2 \\ 1 \end{pmatrix}}_{\vec{a}} + \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{\vec{b}} i$

The general solution is

$$\vec{x} = c_1 \left(e^{-t}\cos t \begin{pmatrix} 2 \\ 1 \end{pmatrix} - e^{-t}\sin t \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) + c_2 \left(e^{-t}\cos t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + e^{-t}\sin t \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right)$$

Howevor this problem has an initial condition $\vec{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ so we need to solve for c_1 & c_2

$\vec{x}(0) = c_1 \left(\begin{pmatrix} 2 \\ 1 \end{pmatrix} \right) + c_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ * $e^0 = 1$ $\sin 0 = 0$ $\cos 0 = 1$

$= \begin{pmatrix} 2c_1 + c_2 \\ c_1 + 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \rightarrow c_1 = 1 \quad c_2 = -1$

$$\vec{x} = e^{-t}\cos t\begin{pmatrix}2\\1\end{pmatrix} - e^{-t}\sin t\begin{pmatrix}1\\0\end{pmatrix} - 1\left(e^{-t}\cos t\begin{pmatrix}1\\0\end{pmatrix} + e^{-t}\sin t\begin{pmatrix}2\\1\end{pmatrix}\right)$$

$$\vec{x} = \begin{pmatrix}2e^{-t}\cos t\\ e^{-t}\cos t\end{pmatrix} - \begin{pmatrix}e^{-t}\sin t\\ 0\end{pmatrix} - \begin{pmatrix}e^{-t}\cos t\\ 0\end{pmatrix} - \begin{pmatrix}2e^{-t}\sin t\\ e^{-t}\sin t\end{pmatrix}$$

$$\boxed{\vec{x} = \begin{pmatrix}e^{-t}\cos t - 3e^{-t}\sin t\\ e^{-t}\cos t - e^{-t}\sin t\end{pmatrix}}$$

EX 2 Find the solution of the given IVP

$$\vec{x}\,' = \begin{pmatrix}-3 & 2\\ -1 & -1\end{pmatrix}\vec{x} \qquad \vec{x}(0) = \begin{pmatrix}1\\ -2\end{pmatrix}$$

* Eigenvalues $\left|\begin{pmatrix}-3 & 2\\ -1 & -1\end{pmatrix} - rI\right| = 0$

$$\begin{vmatrix}-3-r & 2\\ -1 & -1-r\end{vmatrix} = (-3-r)(-1-r) - (-2) = 0$$

$$3 + 3r + r + r^2 + 2 = 0$$

$$r^2 + 4r + 5 = 0$$

$$r = \frac{-4 \pm \sqrt{16-20}}{2} = \frac{-4 \pm \sqrt{-4}}{2} = -2 \pm i$$

$\alpha = -2$, $\beta = 1$

* Eigenvectors

$$(A - rI)\vec{u} = \vec{0}$$

$$(A - (-2+i)I)\vec{u} = 0$$

$$(A + (2-i)I)\vec{u} = \begin{pmatrix}-1-i & 2\\ -1 & 1-i\end{pmatrix}\begin{pmatrix}u_1\\ u_2\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$$

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* Both rows give the same information

$$-u_1 + (1-i)u_2 = 0$$

$$u_1 = (1-i)u_2$$

Let $u_2 = 1 \rightarrow u_1 = 1-i$

$$\vec{u} = \begin{pmatrix} 1-i \\ 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{\vec{a}} + \underbrace{\begin{pmatrix} -1 \\ 0 \end{pmatrix}}_{\vec{b}} i$$

General solution:

$$\vec{x} = c_1\left(e^{-2t}\cos t \begin{pmatrix} 1 \\ 1 \end{pmatrix} - e^{-2t}\sin t \begin{pmatrix} -1 \\ 0 \end{pmatrix}\right)$$

$$+ c_2\left(e^{-2t}\cos t \begin{pmatrix} -1 \\ 0 \end{pmatrix} + e^{-2t}\sin t \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$$

$\vec{x}(0) = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ * Use our initial condition to find c_1 & c_2

$$\vec{x}(0) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$= \begin{pmatrix} c_1 - c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$c_1 - c_2 = 1$

$c_1 = -2 \rightarrow -2 - c_2 = 1$

$c_2 = -3$

$$\vec{x} = -2\left(e^{-2t}\cos t \begin{pmatrix} 1 \\ 1 \end{pmatrix} - e^{-2t}\sin t \begin{pmatrix} -1 \\ 0 \end{pmatrix}\right) - 3\left(e^{-2t}\cos t \begin{pmatrix} -1 \\ 0 \end{pmatrix} + e^{-2t}\sin t \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$$

$$\vec{x} = e^{-2t}\begin{pmatrix} -2\cos t - 2\sin t + 3\cos t - 3\sin t \\ -2\cos t \quad -3\sin t \end{pmatrix}$$

$$\boxed{\vec{x} = e^{-2t}\begin{pmatrix} \cos t - 5\sin t \\ -2\cos t - 3\sin t \end{pmatrix}}$$

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Ex 3 Find the general solution

$$\vec{x}' = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{pmatrix} \vec{x}$$

Eigenvalues $\begin{vmatrix} 1-r & 0 & 0 \\ 2 & 1-r & -2 \\ 3 & 2 & 1-r \end{vmatrix} = (1-r) \begin{vmatrix} 1-r & -2 \\ 2 & 1-r \end{vmatrix}$

$$= (1-r)\left[(1-r)^2 - (-2)(2)\right] = 0$$

$$= (1-r)\left[1 - 2r + r^2 + 4\right] = 0$$

$$= (1-r)(r^2 - 2r + 5)$$

$$r = \frac{2 \pm \sqrt{4-20}}{2} = 1 \pm 2i$$

$\uparrow_{\alpha=1}$ $\uparrow_{\beta=2}$

* 3 eigenvalues $r_1 = 1$ $r_2 = 1 + 2i$ $r_3 = 1 - 2i$

r_1: $(A - rI)\vec{u} = \vec{0}$

$(A - I)\vec{u} = \vec{0}$

$$\begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & -2 \\ 3 & 2 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$2u_1 - 2u_3 = 0$ $u_1 = u_3$

$3u_1 + 2u_2 = 0$

* Both u_2 & u_3 depend on u_1. Let $u_1 = 2$

$u_2 = -\frac{3}{2}u_1$

$$\vec{u}_1 = \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix} s$$

I picked this so we wouldn't be working with fractions

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* Now for the complex eigenvectors

$$(A-(1+2i)I)\vec{u} = \vec{0}$$

$$\begin{pmatrix} -2i & 0 & 0 \\ 2 & -2i & -2 \\ 3 & 2 & -2i \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-2u_1 = 0 \rightarrow u_1 = 0$$

$$\left.\begin{aligned} -2i u_2 - 2u_3 &= 0 \\ 2u_2 - 2i u_3 &= 0 \end{aligned}\right\}$$ Both give same information

$$2u_2 = 2i u_3$$
$$u_2 = i u_3$$

let $u_3 = 1 \rightarrow u_2 = i$

$$\vec{u} = \begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{\vec{a}} + i\underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}_{\vec{b}}$$

$r_1 = 1$ $\alpha = 1$

$$\vec{X} = c_1 e^{r_1 t}\vec{u_1} + c_2\left(e^{\alpha t}\cos\beta t\,\vec{a} - e^{\alpha t}\sin\beta t\,\vec{b}\right) + c\left(e^{\alpha t}\cos\beta t\,\vec{b} + e^{\alpha t}\sin\beta t\,\vec{a}\right)$$

$$\vec{X} = c_1 e^{t}\begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix} + c_2\left(e^{t}\cos 2t\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - e^{t}\sin 2t\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right) + c_3\left(e^{t}\cos 2t\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + e^{t}\sin 2t\right.$$

$$\boxed{\vec{X} = c_1\begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}e^{t} + c_2 e^{t}\begin{pmatrix} 0 \\ -\sin 2t \\ \cos 2t \end{pmatrix} + c_3 e^{t}\begin{pmatrix} 0 \\ \cos 2t \\ \sin 2t \end{pmatrix}}$$

If we were given the initial Condition $\vec{x}(0) = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$

$$\vec{x} = c_1 \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

$$\vec{x}(0) = \begin{pmatrix} 2c_1 \\ -3c_1 + c_3 \\ 2c_1 + c_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

$2c_1 = 2 \longrightarrow c_1 = 1$

$-3c_1 + c_3 = 0 \qquad c_3 = 3$

$2c_1 + c_2 = 1 \qquad c_2 = -1$

$$\vec{x} = e^t \left[\begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix} - \begin{pmatrix} 0 \\ -\sin 2t \\ \cos 2t \end{pmatrix} + 3 \begin{pmatrix} 0 \\ \cos 2t \\ \sin 2t \end{pmatrix} \right]$$

$$\boxed{\vec{x} = e^t \begin{pmatrix} 2 \\ -3 + \sin 2t + 3\cos 2t \\ 2 - \cos 2t + 3\sin 2t \end{pmatrix}}$$