1)

# 12.3 Almost Linear Systems

An autonomous system

$$x' = ax + by + F(x,y)$$
$$y' = cx + dy + G(x,y)$$

is <u>almost</u> <u>linear</u> at the origin if

$$\frac{F(x,y)}{\sqrt{x^2+y^2}} \to 0 \quad \text{and} \quad \frac{G(x,y)}{\sqrt{x^2+y^2}} \to 0$$

as $\sqrt{x^2+y^2} \to 0$. If it is almost linear then we can classify the critical point of the corresponding linear system $(x' = ax + by,\ y' = cx + dy)$ by using the following chart. The eigenvalues of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ are the roots mentioned.

2)

| Roots | Type | Stability (you won't be given this) |
| --- | --- | --- |
| distinct, + | improper node | unstable |
| distinct, − | improper node | asymptotically stable |
| opposite signs | saddle pt | unstable |
| equal, + | improper node or proper node or spiral pt | unstable |
| equal, − | improper node or proper node or spiral pt | asymptotically stable |
| $r = \alpha \pm \beta i, \alpha > 0$ | spiral | unstable |
| $r = \alpha \pm \beta i, \alpha < 0$ | spiral | asymptotically stable |
| $r = \pm \beta i$ | center or spiral | Indeterminate |

Mostly this is the same chart as before, but there are a few changes which I've circled.

Indeterminate because it could be stable (if it is a center) or unstable or asymptotically stable (if it is a spiral)

3)

Show that the given system is almost linear near the origin & discuss the type & stability of the critical pt at the origin

1. $\frac{dx}{dt} = 3x + 2y - y^2$

$\frac{dy}{dt} = -2x - 2y + xy$

☆ Note: they aren't asking us to find & classify all critical pts, just the one at the origin.

First we'll show it is almost linear:

$F(x,y) = -y^2$

$G(x,y) = xy$

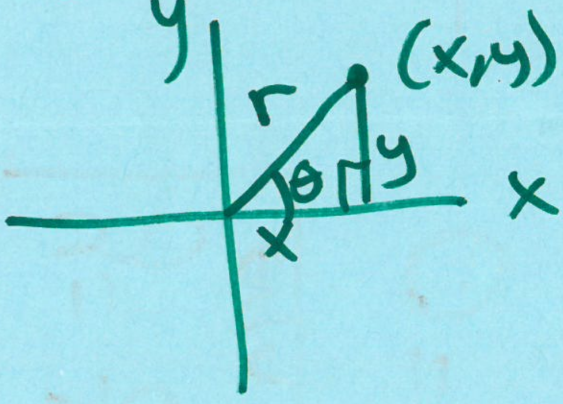
4) We need to show

$$\frac{F}{\sqrt{x^2+y^2}} \to 0 \quad \text{as} \quad \sqrt{x^2+y^2} \to 0$$

Usual technique for doing this is to switch to polar coordinates

Recall! In polar coordinates (used in Calc 3):

$$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ x^2+y^2 &= r^2 \end{aligned}}$$




r is the distance from the origin to our point.

$$\frac{F}{\sqrt{x^2+y^2}} = \frac{-y^2}{\sqrt{x^2+y^2}} = \frac{-(r\sin\theta)^2}{\sqrt{r^2}} = \frac{-r^2\sin^2\theta}{r}$$

$$= -r\sin^2\theta$$

$\sqrt{x^2+y^2} \to 0$ is the same as

$\sqrt{r^2} \to 0$ or $r \to 0^+$

5) $\lim_{r\to 0^+} -r\sin^2\theta = 0$ ✓

We need to show $\frac{G}{\sqrt{x^2+y^2}} \to 0$ as well

$$\frac{G}{\sqrt{x^2+y^2}} = \frac{xy}{\sqrt{x^2+y^2}} = \frac{\cancel{r}\cos\theta\, r\sin\theta}{\cancel{\sqrt{r^2}}} \to 0$$

as $r \to 0$ ✓

So this is almost linear

To classify the critical point at the origin, set $F=G=0$ to get

$$\frac{dx}{dt} = 3x + 2y$$

$$\frac{dy}{dt} = -2x - 2y$$

$$|A - rI| = \begin{vmatrix} 3-r & 2 \\ -2 & -2-r \end{vmatrix} = (3-r)(-2-r)+4 =$$

$$r^2 - r - 2 = 0$$

$$(r-2)(r+1) = 0$$

6) $r_1 = 2$, $r_2 = -1$

$\boxed{(0,0) \text{ is an unstable saddle}}$

Find all the critical points & classify them

9. $\frac{dx}{dt} = 16 - xy$

$\frac{dy}{dt} = x - y^3$

Set $\frac{dx}{dt} = 0$ & $\frac{dy}{dt} = 0$ to find critical points

$16 - xy = 0$

$x - y^3 = 0 \longrightarrow x = y^3$

$16 - xy = 0$

$16 - y^3 y = 0$

$16 - y^4 = 0 \longleftrightarrow (4 - y^2)(4 + y^2) = 0$

7) $(4-y^2)(4+y^2)=0$ Fun w/ difference of Squares!

$(2+y)(2-y)(4+y^2)=0$

$y=-2,\ y=2$ $\quad 4+y^2 \neq 0$ for real valued y

If $y=-2$, $x=(-2)^3=-8$

If $y=2$, $x=(2)^3=8$

Two critical points $(-8,-2)$ & $(8,2)$.

☆ Since neither critical point is at the origin we'll need to shift both of them.

Starting w/ $(-8,-2)$

$x=u+a \qquad y=v+b$

$x=u-8 \qquad y=v-2$

$\frac{dx}{dt}=\frac{du}{dt} \qquad \frac{dy}{dt}=\frac{dv}{dt}$

8) Plugging into our d.e

$$\frac{du}{dt} = 16 - (u-8)(v-2)$$

$$= 16 - [uv - 8v - 2u + 16]$$

$$= -uv + 8v + 2u$$

$$= 2u + 8v \underbrace{- uv}_{F}$$

$$\frac{dv}{dt} = (u-8) - (v-2)^3$$

$$= u - 8 - (v-2)(v^2 - 4v + 4)$$

$$= u - 8 - [v^3 - \underline{4v^2} + 4v - \underline{2v^2} + 8v - 8]$$

$$= u - \cancel{8} - v^3 + 6v^2 - 12v + \cancel{8}$$

$$= u - 12v \underbrace{- v^3 + 6v^2}_{G}$$

If we wanted to show its almost linear at (-8, -2) we would use this F & G

9)

$$A = \begin{bmatrix} 2 & 8 \\ 1 & -12 \end{bmatrix}$$

$$|A - rI| = \begin{vmatrix} 2-r & 8 \\ 1 & -12-r \end{vmatrix} =$$

$$(2-r)(-12-r) - 8 =$$

$$-24 + 10r + r^2 - 8 =$$

$$r^2 + 10r - 32 = 0$$

$$r = \frac{-10 \pm \sqrt{100 - 4(-32)}}{2}$$

positive number under the sqrt

$$= \frac{-10 \pm \sqrt{228}}{2}$$

$r_1 > 0,\ r_2 < 0$

$\boxed{(-8,-2) \text{ unstable saddle}}$

10) We need to do the same thing for our other critical point

$x = u + 8 \quad y = v + 2$

$$\frac{du}{dt} = 16 - (u+8)(v+2)$$

$$= 16 - uv - 8v - 2u - 16$$

$$= \underbrace{-2}_{a}u \underbrace{-8}_{b}v \underbrace{-uv}_{F}$$

$$\frac{dv}{dt} = (u+8) - (v+2)^3$$

$$= u + \cancel{8} - (v^3 + 6v^2 + 12v + \cancel{8})$$

$$= \underbrace{1}_{c}u \underbrace{-12}_{d}v \underbrace{-v^3 - 6v^2}_{G}$$

used Pascal's triangle on a separate sheet of paper -- took longer than just multiplying it out

11)
$$A = \begin{bmatrix} -2 & -8 \\ 1 & -12 \end{bmatrix}$$

$$|A - rI| = \begin{vmatrix} -2-r & -8 \\ 1 & -12-r \end{vmatrix}$$

$$= (-2-r)(-12-r) + 8$$

$$= r^2 + 14r + 24 + 8$$

$$= r^2 + 14r + 32 = 0$$

$$r = \frac{-14 \pm \sqrt{14^2 - 4(32)}}{2}$$

$$= \frac{-14 \pm \sqrt{2^2 \cdot 7^2 - 4(32)}}{2}$$

$$= \frac{-14 \pm \sqrt{4 \cdot 49 - 4(32)}}{2}$$

$$= \frac{-14 \pm \sqrt{4(49 - 32)}}{2}$$

I forgot that calculators exist

12)

$$r = \frac{-14 \pm \sqrt{4(17)}}{2}$$

$$= \frac{-14 \pm 2\sqrt{17}}{2}$$

$$= -7 \pm \sqrt{17}$$

$$r_1 = -7 + \underbrace{\sqrt{17}}_{\text{little bigger than } 4}, \quad r_2 = -7 - \sqrt{17}$$

Whole point of this, is that

$r_1 < 0$ & $r_2 < 0$

$\boxed{\text{(8,2) is therefore an asymptotically stable improper node}}$