1)

Arc Length

In class we derived that the length of a smooth curve given by parametric equations

$$x = x(t)$$
$$y = y(t)$$
$$a \leq t \leq b$$

is

$$\boxed{L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt}$$

Alternatively, if we have a curve given by $y = f(x)$ with $a \leq x \leq b$ we'd get

$$\boxed{L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx}$$

2) We'd get that by making the substitutions
$x = x \quad y = f(x)$ into the 1st formula.

Likewise if C is given by $x = g(y)$, $c \le y \le d$

$$L = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy$$

Many of the following examples come from Calculus by Sullivan & Miranda.

3)

Ex 1: $x = t^3 \quad y = \frac{2}{3}t^{\frac{9}{2}}$

$t=0$ to $t=1$ find the length of the curve.

☆ Parametric eqns so we use $L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$$= \int_0^1 \sqrt{(3t^2)^2 + \left(\frac{\cancel{2}}{3}\cdot\frac{9}{\cancel{2}} t^{7/2}\right)^2}\, dt$$

$$= \int_0^1 \sqrt{9t^4 + (3t^{7/2})^2}\, dt$$

don't forget to square the 3!

4)
$$\int_0^1 \sqrt{9t^4 + 9t^7}\, dt$$

$$= \int_0^1 \sqrt{\underbrace{9t^4}_{}(1+t^3)}\, dt$$

pull out common factor

$$= \int_0^1 \underline{3t^2}\sqrt{1+t^3}\, \underline{dt}$$

$u = 1+t^3$ u-substitution!

$du = \underline{3t^2 dt}$ $\quad u(0) = 1+0^3 = 1$

$u(1) = 1+1^3 = 2$

$$= \int_1^2 \sqrt{u}\, du$$

$$= \int_1^2 u^{1/2}\, du$$

$$= \frac{2}{3}u^{3/2}\Big|_1^2 = \frac{2}{3}\cdot 2^{3/2} - \frac{2}{3}\cdot 1^{3/2}$$

$$= \boxed{\frac{2}{3}\cdot 2^{3/2} - \frac{2}{3}}$$

5) Ex 2:

$$y = \frac{2}{9}\sqrt{3}\left(3x^2+1\right)^{3/2}$$ from $x=-1$ to $x=2$

Because we have a $y=$ formula, we'll use

$$L = \int_a^b \sqrt{1+[f'(x)]^2}\, dx$$

$$= \int_{-1}^{2} \sqrt{1+\Big[\underbrace{\tfrac{\cancel{2}}{9}\sqrt{3}\,\tfrac{3}{\cancel{2}}\left(3x^2+1\right)^{1/2} 6x}_{\text{fun with chain rule}}\Big]^2}\, dx$$

$$= \int_{-1}^{2} \sqrt{1+\left[2\sqrt{3}\left(3x^2+1\right)^{1/2}x\right]^2}\, dx$$

$$= \int_{-1}^{2} \sqrt{1+4\cdot 3\left(3x^2+1\right)x^2}\, dx$$

6) $\int_{-1}^{2} \sqrt{1+(36x^2+12)x^2}\, dx$

$= \int_{-1}^{2} \sqrt{\underbrace{1+36x^4+12x^2}}\, dx$

3 terms, try to factor

Recall: $(a+b)^2 = a^2+2ab+b^2$

we want to write

$1+36x^4+12x^2 = (\quad)^2$

if possible. So we need to look for what would be the a^2 & b^2 terms

$36x^4$ looks like a good candidate for $a^2 \rightarrow a = 6x^2$

($-6x^2$ is a possibility but based on our signs we can rule that out)

7)

$$1+36x^4+12x^2=(6x^2+1)^2$$

I decided 1 was a better candidate for b^2 based on the numbers they gave us. Remember after we factor, we need to check to see if it works

$$(6x^2+1)^2=36x^4+2(6x^2)(1)+1^2 \checkmark$$

$$\int_{-1}^{2}\sqrt{(6x^2+1)^2}\,dx$$

$$=\int_{-1}^{2}6x^2+1\,dx = 2x^3+x\Big|_{-1}^{2}$$

$$=16+2-[2(-1)^3-1]$$

$$=18+2+1=\boxed{21}$$

8) Ex 3:

Find the length of the curve $9y^2=4x^3$ from $x=0$ to $x=1$; $y\geq 0$

☆ This problem has us choosing to solve for x or y. Based on their stipulation that $y\geq 0$ & that they gave x bounds, I'd solve for y:

$$y^2 = \frac{4}{9}x^3$$

$$y = \pm\sqrt{\frac{4}{9}x^3} = \pm\frac{2}{3}x^{3/2}$$

$y\geq 0$ so $y = \frac{2}{3}x^{3/2}$

9)
$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

$$= \int_0^1 \sqrt{1 + \left[\frac{2}{3} \cdot \frac{3}{2} x^{1/2}\right]^2}\, dx$$

$$= \int_0^1 \sqrt{1 + [x^{1/2}]^2}\, dx$$

$$= \int_0^1 \sqrt{1 + x}\, dx$$

$u = 1 + x$ $\quad u(0) = 1 + 0 = 1$

$du = dx$ $\quad u(1) = 1 + 1 = 2$

$$= \int_1^2 \sqrt{u}\, du = \int_1^2 u^{1/2}\, du$$

$$\frac{2}{3} u^{3/2} \Big|_1^2 = \boxed{\frac{2}{3} \cdot \left[2^{3/2} - 1\right]}$$

$1^{3/2}$

☆ This one wasn't that exciting

10)

Ex 4: $e^x = \cos y$

from $\left(\ln\frac{\sqrt{3}}{2}, \frac{\pi}{6}\right)$ to $\left(\ln\frac{1}{2}, \frac{\pi}{3}\right)$

Let's get this in terms of $x=$, since the y values of our points look nicer

$$e^x = \cos y$$

$$\ln(e^x) = \ln(\cos y)$$

$$x = \ln(\cos y)$$

$$L = \int_c^d \sqrt{1 + [f'(y)]^2}\, dy$$

$$= \int_{\pi/6}^{\pi/3} \sqrt{1 + \left[\frac{1}{\cos y}(\sin y)\right]^2}\, dy$$

11) $L = \int_{\pi/6}^{\pi/3} \sqrt{1 + \frac{\sin^2 y}{\cos^2 y}}\, dy$

$= \int_{\pi/6}^{\pi/3} \sqrt{\underbrace{\frac{\cos^2 y}{\cos^2 y}}_{1} + \frac{\sin^2 y}{\cos^2 y}}\, dy$

$= \int_{\pi/6}^{\pi/3} \sqrt{\frac{\cos^2 y + \sin^2 y}{\cos^2 y}}\, dy$

$= \int_{\pi/6}^{\pi/3} \sqrt{\frac{1}{\cos^2 y}}\, dy$

$= \int_{\pi/6}^{\pi/3} \frac{1}{\cos y}\, dy$

$= \int_{\pi/6}^{\pi/3} \sec y\, dy$

12) $\int_{\pi/6}^{\pi/3} \sec y \cdot 1 \, dy$

$\overset{\text{tricky}}{=} \int_{\pi/6}^{\pi/3} \sec y \cdot \frac{\sec y + \tan y}{\sec y + \tan y} \, dy$

$= \int_{\pi/6}^{\pi/3} \frac{\sec^2 y + \sec y \tan y}{\sec y + \tan y} \, dy$

$u = \sec y + \tan y$

$du = (\sec y \tan y + \sec^2 y) \, dy$

$= \int \frac{1}{u} \, du$

$u(\pi/3) = \sec\frac{\pi}{3} + \tan\frac{\pi}{3} = \frac{1}{1/2} + \frac{\sqrt{3}/2}{1/2} = 2 + \sqrt{3}$

$u(\pi/6) = \sec\frac{\pi}{6} + \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}/2} + \frac{1/2}{\sqrt{3}/2} = \frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}}$

$= 3/\sqrt{3}$

13)

$$\int_{\sqrt{3}}^{2+\sqrt{3}} \frac{1}{u}\,du$$

$$\frac{3}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{3\sqrt{3}}{3} = \sqrt{3}$$

$$= \ln|u| \Big|_{\sqrt{3}}^{2+\sqrt{3}}$$

$$= \boxed{\ln(2+\sqrt{3}) - \ln\sqrt{3}}$$