:) The Binomial Series

* In class we derived the Maclaurin series for $(1+x)^k$ we found

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)x^2}{2!} + \frac{k(k-1)(k-2)x^3}{3!} + \cdots$$

when $|x|<1$

this is our Binomial Series

Examples from Calculus by Strauss, Bradley, Smith p 564

Use the Binomial series to expand the function as a power series. State the radius of convergence.

Ex 1 $(4+x)^{-1/3} = f(x)$

* We need to match the form

$$(1+(\ ))^k = \sum_{n=0}^{\infty} \binom{k}{n} (\ )^n = 1 + k(\ ) + \frac{k(k-1)(\ )^2}{2!} + \cdots$$

$$(4+x)^{-1/3} = \left(4\left(1+\frac{x}{4}\right)\right)^{-1/3} = 4^{-1/3}\left(1+\frac{x}{4}\right)^{-1/3}$$

$$= 4^{-1/3} \sum_{n=0}^{\infty} \binom{-\frac{1}{3}}{n} \left(\frac{x}{4}\right)^n$$

* It is easier to leave on the outside

$\left|\frac{x}{4}\right| < 1 \quad R=4$

P2

$$(4+x)^{-1/3} = 4^{-1/3}\sum_{n=0}^{\infty}\binom{-\frac{1}{3}}{n}\left(\frac{x}{4}\right)^n$$ *Now we expand it*

$$= 4^{-1/3}\left[1 - \frac{1}{3}\left(\frac{x}{4}\right) - \frac{\frac{1}{3}\left(-\frac{1}{3}-1\right)\left(\frac{x}{4}\right)^2}{2!} - \frac{\frac{1}{3}\left(-\frac{1}{3}-1\right)\left(-\frac{1}{3}-2\right)\left(\frac{x}{4}\right)^3}{3!} - \cdots\right]$$

$$= 4^{-1/3}\left[1 - \frac{1}{3}\left(\frac{x}{4}\right) - \frac{\frac{1}{3}\left(-\frac{4}{3}\right)\left(\frac{x^2}{4^2}\right)}{2!} - \frac{\frac{1}{3}\left(-\frac{4}{3}\right)\left(-\frac{7}{3}\right)\left(\frac{x^3}{4^3}\right)}{3!} - \cdots\right]$$

* Describe the easy things first while keeping in mind that your series might not start at zero because of this I leave room*

$$= 4^{-1/3}\left[\frac{?}{\quad} + \frac{?}{\quad} + \sum_{n=}^{\infty}\frac{x^n(-1)^n(\ )\cdot(\ )\cdot(\ )\cdots(\ )}{4^n 3^n n!}\right]$$

We can fill in the blanks by using

$$K-n+1 = -\frac{1}{3}-n+1 = \frac{2}{3}-n$$

We have already accounted for the $\frac{1}{3}$ with our $\frac{1}{3^n}$ so $\frac{2}{3}-n = \frac{1}{3}(2-3n)$

We also have already dealt with the negative so $\frac{2}{3}-n = \frac{1}{3}(2-3n) = -\frac{1}{3}(3n-2)$

3n-2 goes into our last (.)

We can determine where to start our series by looking when $3n-2 > 0$

Test: $n=0 \quad 3(0)-2=-2<0$ ✗

$n=1 \quad 3(1)-2=1>0$ ✓

We need to start the sum at 1

$$=4^{-1/3}\left[1+\sum_{n=1}^{\infty}\frac{x^n(-1)^n(1)(3(2)-2)(3(3)-2)\cdots(3n-2)}{4^n 3^n n!}\right]$$

$$=4^{-1/3}\left[1+\sum_{n=1}^{\infty}\frac{x^n(-1)^n 1\cdot 4\cdot 7\cdots(3n-2)}{4^n 3^n n!}\right]$$

$$=\boxed{4^{-1/3}+\sum_{n=1}^{\infty}\frac{x^n(-1)^n 1\cdot 4\cdot 7\cdots(3n-2)}{4^{n+1/3}\, 3^n\, n!}}$$

Ex 2 $\quad f(x)=\frac{x}{\sqrt{1-x^2}}=x(1-x^2)^{-1/2}=x(1+(-x^2))^{-1/2}$

leave on outside

$$=x\sum_{n=0}^{\infty}\binom{-1/2}{n}(-x^2)^n$$

$|-x^2|<1 \rightarrow R=1$

Expand

$$=x\left[1-\frac{1}{2}(-x^2)-\frac{\frac{1}{2}(-\frac{1}{2}-1)(-x^2)^2}{2!}-\frac{\frac{1}{2}(-\frac{1}{2}-1)(-\frac{1}{2}-2)(-x^2)^3}{3!}\cdots\right]$$

P4

$$= x\left[1 - \frac{1}{2}(-x^2) - \frac{\frac{1}{2}\left(\frac{-3}{2}\right)x^4}{2!} - \frac{\frac{1}{2}\left(\frac{-3}{2}\right)\left(\frac{-5}{2}\right)(-x^6)}{3!} + \cdots\right]$$

$$= x\left[1 + \frac{1}{2}x^2 + \frac{3x^4}{2^2\,2!} + \frac{3\cdot 5\,x^6}{2^3\,3!} + \cdots\right]$$

$$= x\left[\underline{?} + \underline{?} + \sum_{n=}^{\infty} \frac{x^{2n}(\ )\cdot(\ )\cdot(\ )\cdots(\ )}{2^n\,n!}\right]$$

$$k-n+1 = -\frac{1}{2}-n+1 = \frac{1}{2}-n = \frac{1}{2}(1-2n) = -\frac{1}{2}(2n-1)$$

$n=0 \quad 2(0)-1 = -1 < 0$ X can't start at $n=0$

$n=1 \quad 2(1)-1 = 1 > 0$

$n=2 \quad 2(2)-1 = 3$

$n=3 \quad 2(3)-1 = 5$

$$\frac{x}{\sqrt{1-x^2}} = x\left[1 + \sum_{n=1}^{\infty} \frac{x^{2n}\,(1)\cdot 3\cdot 5\cdots(2n-1)}{2^n\,n!}\right]$$

$$= \boxed{x + \sum_{n=1}^{\infty} \frac{x^{2n+1}\,1\cdot 3\cdot 5\cdots(2n-1)}{2^n\,n!}}$$

P5

Ex3 $f(x) = \sqrt[4]{2-x}$

$$\sqrt[4]{2-x} = (2-x)^{1/4} = \left(2\left(1-\frac{x}{2}\right)\right)^{1/4} = 2^{1/4}\left(1+\left(-\frac{x}{2}\right)\right)^{1/4}$$

$$= 2^{1/4} \sum_{n=0}^{\infty} \binom{1/4}{n} \left(-\frac{x}{2}\right)^n \qquad \left|\frac{-x}{2}\right| < 1 \rightarrow R = 2$$

$$= 2^{1/4}\left[1 + \frac{1}{4}\left(\frac{-x}{2}\right) + \frac{\frac{1}{4}\left(\frac{1}{4}-1\right)\left(\frac{-x}{2}\right)^2}{2!} + \frac{\frac{1}{4}\left(\frac{1}{4}-1\right)\left(\frac{1}{4}-2\right)\left(\frac{-x}{2}\right)^3}{3!} + \cdots\right]$$

$$= 2^{1/4}\left[1 + \frac{1}{4}\left(-\frac{x}{2}\right) + \frac{\frac{1}{4}\left(-\frac{3}{4}\right)\left(\frac{x^2}{2^2}\right)}{2!} + \frac{\frac{1}{4}\left(\frac{-3}{4}\right)\left(\frac{-7}{4}\right)\left(\frac{-x^3}{2^3}\right)}{3!} + \cdots\right]$$

* Notice this series is not alternating. Every term except the first will be negative. We know then that the first term is definitely not included in Σ.

$$= 2^{1/4}\left[1 + \underline{\ ?\ } + \sum^{\infty} \frac{-x^n (\ )\cdot(\ )\cdot(\ )\cdots(\quad)}{4^n 2^n n!}\right]$$

$$k - n + 1 = \frac{1}{4} - n + 1 = \frac{5}{4} - n = \frac{1}{4}(5-4n)$$
$$= -\frac{1}{4}(4n-5)$$

P6

$n=0 \quad 4(0)-5=-5<0$ ✗ we already knew we couldn't start at $n=0$

$n=1 \quad 4(1)-5=-1<0$ ✗

$n=2 \quad 4(2)-5=3>0$ ✓ starting at $n=2$

$n=3 \quad 4(3)-5=7$

$n=4 \quad 4\cdot 4-5=11$

$$\sqrt[4]{2-x}=2^{1/4}\left[1+\frac{1}{4}\left(\frac{-x}{2}\right)+\sum_{n=2}^{\infty}\frac{-x^n\,3\cdot 7\cdot 11\cdots(5-4n)}{4^n 2^n n!}\right]$$

$$=2^{1/4}+\frac{(-x)2^{1/4}}{\underbrace{4}_{2^2}\cdot 2}+\sum_{n=2}^{\infty}\frac{-x^n\,3\cdot 7\cdot 11\cdots(5-4n)\,2^{1/4-n}}{4^n\, n!}$$

$$=2^{1/4}-2^{\frac{1}{4}-3}x+\sum_{n=2}^{\infty}\frac{-x^n\,3\cdot 7\cdot 11\cdots(5-4n)\,2^{\frac{1}{4}-n}}{4^n\, n!}$$

$$=\boxed{2^{1/4}-2^{-11/4}x+\sum_{n=2}^{\infty}\frac{-x^n\,3\cdot 7\cdot 11\cdots(5-4n)\,2^{\frac{1}{4}-n}}{4^n\, n!}}$$

Ex 4

$$f(x) = \sqrt[5]{7+x}$$
$$= (7+x)^{1/5}$$
$$= \left(7\left(1+\frac{x}{7}\right)\right)^{1/5}$$
$$= 7^{1/5}\left(1+\frac{x}{7}\right)^{1/5}$$
$$= 7^{1/5}\sum_{n=0}^{\infty}\binom{\frac{1}{5}}{n}\left(\frac{x}{7}\right)^n$$
$$= 7^{1/5}\left[1 + \frac{1}{5}\left(\frac{x}{7}\right) + \frac{\frac{1}{5}\left(\frac{1}{5}-1\right)\left(\frac{x}{7}\right)^2}{2!} + \frac{\frac{1}{5}\left(\frac{1}{5}-1\right)\left(\frac{1}{5}-2\right)\left(\frac{x}{7}\right)^3}{3!} + \cdots\right]$$
$$= 7^{1/5}\left[1 + \frac{1}{5}\left(\frac{x}{7}\right) + \frac{\frac{1}{5}\left(\frac{-4}{5}\right)\left(\frac{x}{7}\right)^2}{2!} + \frac{\frac{1}{5}\left(\frac{-4}{5}\right)\left(\frac{-9}{5}\right)\left(\frac{x}{7}\right)^3}{3!} + \cdots\right]$$
$$= 7^{1/5}\left[\underline{\ ?\ } + \underline{\ ?\ } + \sum_{n=?}^{\infty}\frac{(-1)^{n+1}x^n\left(()\cdot()\cdot()\cdots()\right)}{5^n n!\, 7^n}\right]$$

$$k-n+1 = \frac{1}{5}-n+1 = \frac{6}{5}-n = \frac{1}{5}(6-5n)$$
$$= -\frac{1}{5}(5n-6)$$

$n=0$: $5\cdot 0-6<0$ $\times$ this didn't fit our $(-1)^{n+1}$ pattern anyway

$n=1$: $5\cdot 1-6<0$ $\times$

$n=2$: $5\cdot 2-6>0$ $\checkmark$

$$= 7^{1/5}\left[1 + \frac{1}{5}\left(\frac{x}{7}\right) + \sum_{n=2}^{\infty}\frac{(-1)^{n+1}x^n\left(\overset{5\cdot2-6}{4}\cdot\overset{5\cdot3-6}{9}\cdot\overset{5\cdot4-6}{14}\cdots(5n-6)\right)}{5^n 7^n n!}\right]$$

P8

$$7^{1/5} = \frac{1}{7^{-1/5}}$$

$$\frac{1}{7^{-1/5}}\left[1 + \frac{1}{5}\left(\frac{x}{7}\right) + \sum_{n=2}^{\infty} \frac{(-1)^{n+1} x^n \left(4\cdot 9\cdot 14 \cdots (5n-6)\right)}{5^n 7^n n!}\right]$$

$$= \left[\frac{1}{7^{-1/5}} + \frac{x}{5\cdot 7^{1-1/5}} + \sum_{n=2}^{\infty} \frac{(-1)^{n+1} x^n \left(4\cdot 9\cdot 14 \cdots (5n-6)\right)}{5^n 7^{n-1/5} n!}\right]$$

$$= \boxed{7^{1/5} + \frac{x}{5\cdot 7^{4/5}} + \sum_{n=2}^{\infty} \frac{(-1)^{n+1} x^n \left(4\cdot 9\cdot 14 \cdots (5n-6)\right)}{5^n 7^{n-1/5} n!}}$$