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Derivatives & the Shapes of Curves

We can use derivatives to help us graph functions by finding where they are increasing/decreasing, concave up/down and where they have maxs/mins.

Technique: (1) Find the domain of f.

This is importent because critical numbers only happen in the domain of f. Also, we want to know where our graphs start & end & if they have any vertical asymptotes.

(2) Find f′ and set it equal to zero, solve for x. Find all critical numbers (values of x in the domain of f such that $f'(x)=0$ or $f'(x)=DNE$)

(3) To find where f is increasing or decreasing, we need to know where $f'>0$ and $f'<0$. Form intervals using the critical numbers and vertical asymptotes. Test f′ at values of x in those intervals.

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④ Identify local maxs & mins.

* Local maxs/mins must be in the domain of f (Vertical asymptotes can't be local maxs/mins)

- A local max happens when f goes from increasing to decreasing. /\
- A local min happens when f goes from decreasing to increasing \/

⑤ Find f" & set it equal to zero & solve for X. Form intervals with these values of x & any x where f''(x) = DNE and vertical asymptotes of f.

⑥ Test concavity. When f" > 0, f is concave upwards. When f" < 0, f is concave downwards. Test values of x within these intervals (You are plugging into f")

⑦ Identify inflection points (where we change concavity). Inflection points must be in f's domain!

⑧ Draw f. Find f at any maxs/mins & use your results to get a rough graph of f.

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Examples from Single Variable Calculus
by Strauss, Bradley, and Smith

Ex 1 $f(x)=\frac{1}{3}x^3-9x+2$

① Domain: $\mathbb{R}$

② $f'=x^2-9=0$

$\boxed{x=3,\ x=-3}$ } critical numbers
f' exists for all x

③

Intervals	$f'=x^2-9$	f
$(-\infty,-3)$	$f'(-4)=(-4)^2-9>0$	increasing
$(-3,3)$	$f'(0)=0^2-9<0$	decreasing
$(3,\infty)$	$f'(4)=4^2-9>0$	increasing

For the intervals plug any value of x (except the endpoints) into f'. If $f'>0$, f is increasing. $f'<0$, f is decreasing

④ At $x=-3$ we go from inc to dec

$\boxed{x=-3 \text{ local max}}$

At $x=3$ we go from dec to inc

$\boxed{x=3 \text{ local min}}$

⑤ $f''=2x=0 \quad x=0$

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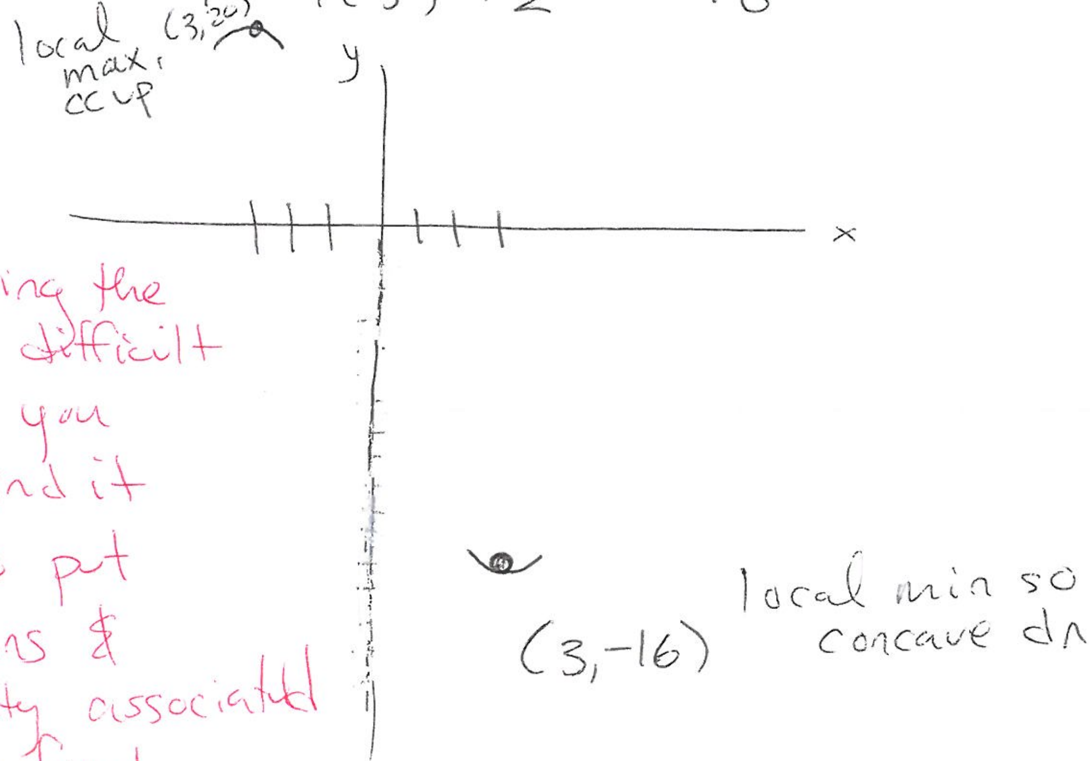
⑥

Intervals	$f'' = 2x$	f
$(-\infty, 0)$	$f''(-1) = -2 < 0$	concave downward
$(0, \infty)$	$f''(1) = 2 > 0$	concave upward

⑦ $\boxed{x = 0 \text{ inflection pt}}$

⑧
$$f(-3) = \tfrac{1}{3}(-3)^3 - 9(-3) + 2$$
$$= -9 + 27 + 2 = 20$$
$$f(3) = \tfrac{1}{3}(3)^3 - 9(3) + 2$$
$$= 9 - 9(3) + 2 = -16$$



* If drawing the graphs is difficult for you, you might find it helpful to put maxs & mins & the concavity associated with them first.

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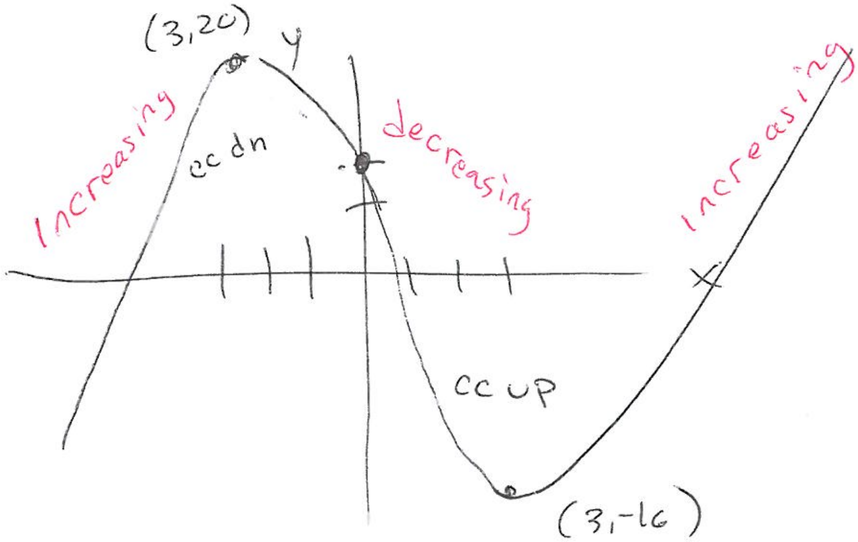

$f(0) = \frac{1}{3}\cdot 0^3 - 9\cdot 0 + 2 = 2$

$\boxed{\text{Ex 2}}$ $f(x) = x + \frac{1}{x}$

① Domain: $x \neq 0$ ($x=0$ vertical asymptote)

② $f'(x) = 1 - x^{-2} = 0$

$1 = x^{-2}$

$1 = \frac{1}{x^2}$

$x^2 = 1$

$\boxed{x = \pm 1}$ critical numbers

Note: $f'(0) = DNE$ but 0 is not in f's domain so it is not a critical number

③

Intervals	$f' = 1 - \frac{1}{x^2}$	f
$(-\infty, -1)$	$f'(-2) = 1 - \frac{1}{(-2)^2} > 0$	inc
$(-1, 0)$	$f'(-\frac{1}{2}) < 0$	dec
$(0, 1)$	$f'(\frac{1}{2}) < 0$	dec
$(1, \infty)$	$f'(2) > 0$	inc

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Note! If you said f is decreasing from (-1, 1) you would be wrong. f(0) = DNE so it can't be included in the interval.

(4) local max $x = -1$ (inc to dec)
local min $x = 1$ (dec to inc)

(5) $f'' = 2x^{-3} = \frac{2}{x^3}$

(6)

Intervals	$f'' = \frac{2}{x^3}$	f
$(-\infty, 0)$	$f''(-1) < 0$	cc dn
$(0, \infty)$	$f''(1) > 0$	cc up

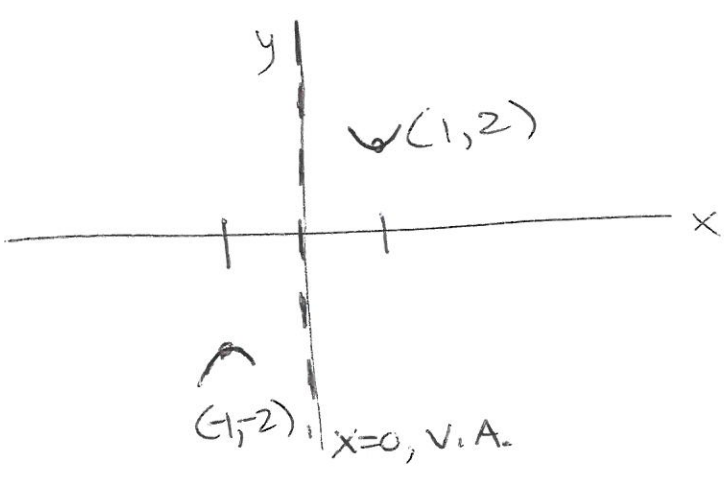
using it in interval since it is not in f's domain
also we'd use it even if it was in f's domain
since f'' is not defined here.

(7) No inflection pts. f changes concavity at $x=0$, but it is not in f's domain.

(8)

$f(-1) = -1 + \frac{1}{-1} = -2$ local max

$f(1) = 1 + \frac{1}{1} = 2$ local min



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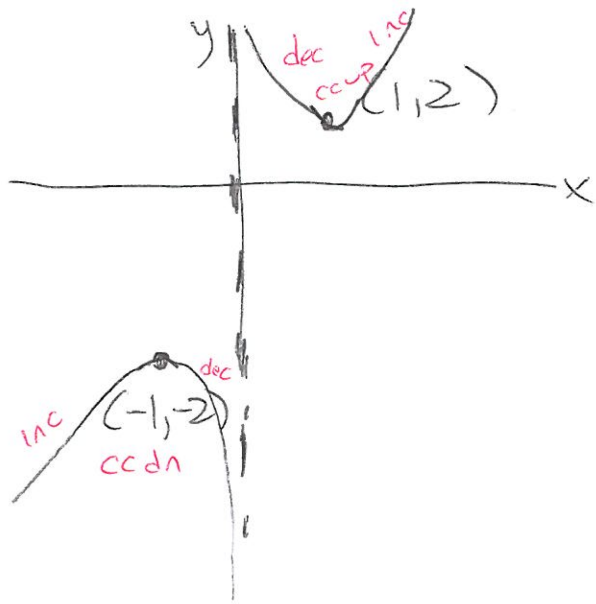

Ex 3 $\quad f(t) = t - \ln t$

① Domain: $t > 0$ ($t=0$ vertical asymptote)

② $f'(t) = 1 - \frac{1}{t} = 0$

$\boxed{t=1}$ critical number

$t=0$ is not a critical number since it is not in f's domain

③

Intervals	$f' = 1 - \frac{1}{t}$	f
$(0,1)$	$f'(\frac{1}{2}) = 1 - \frac{1}{\frac{1}{2}} < 0$	dec
$(1,\infty)$	$f'(2) > 0$	inc

can't start at $(-\infty, \ldots)$ since domain $t>0$

④ $t=1$ local min (dec to inc)

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(5) $f' = 1 - \frac{1}{t} = 1 - t^{-1}$

$f'' = t^{-2} = \frac{1}{t^2}$

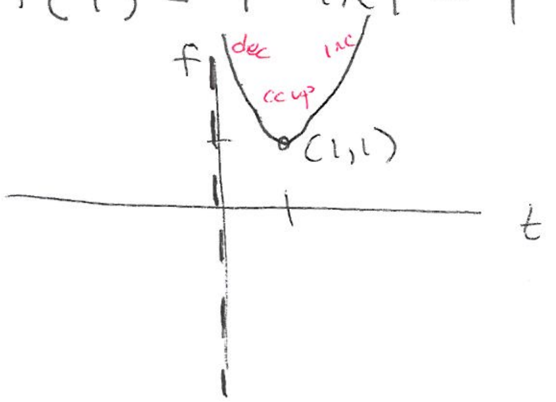
f'' is never zero., $t=0$ only problem spot

(6)

Interval	$f'' = \frac{1}{t^2}$	f
$(0,\infty)$	$f''(1) > 0$	cc up

(7) No inflection pts

(8) $f(1) = 1 - \ln 1 = 1 - 0 = 1$



$t=0$ VA.

$\boxed{\text{Ex 4}}$ Use $f(x) = \frac{x}{x^2+1}$, $f'(x) = \frac{1-x^2}{(x^2+1)^2}$, $f''(x) = \frac{2x(x^2-3)}{(x^2+1)^3}$

to answer the following

a) What is the domain of f?

b) Identify any vertical or horizontal asymptotes.

c) Identify all critical numbers

d) On what intervals is f inc/dec?

e) Identify all local maxs/mins

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f) On what intervals is f concave up/down?

g) Identify all inflection pts

h) Graph f. Label any asymptotes, critical numbers, & find the y-intercept

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a) Domain = $\mathbb{R}$ (denominator never 0)

b) V.A. happen where f is undefined but f is defined for all x (no V.A.)

For Horizontal asymptotes $\lim_{x\to\infty} \frac{x}{x^2+1} = \lim_{x\to\infty} \frac{1}{2x} = 0$

$$\underbrace{\lim_{x\to-\infty} \frac{x}{x^2+1} = \lim_{x\to\infty} \frac{1}{2x} = 0}_{\text{L'Hospital's Rule}}$$

$\boxed{y=0 \text{ H.A.}}$

c) $f' = \frac{1-x^2}{(x^2+1)^2} = 0$ if $1-x^2=0$

$\boxed{x=\pm 1}$ critical numbers

d)

Intervals	$f' = \frac{1-x^2}{(x^2+1)^2}$	f
$(-\infty, -1)$	$f'(-2) < 0$	dec
$(-1, 1)$	$f'(0) > 0$	inc
$(1, \infty)$	$f'(2) < 0$	dec

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e) $x=-1$ local min

$x=1$ local max

f) $f'' = \dfrac{2x(x^2-3)}{(x^2+1)^3} = 0$

$x=0, \ x=\pm\sqrt{3}$

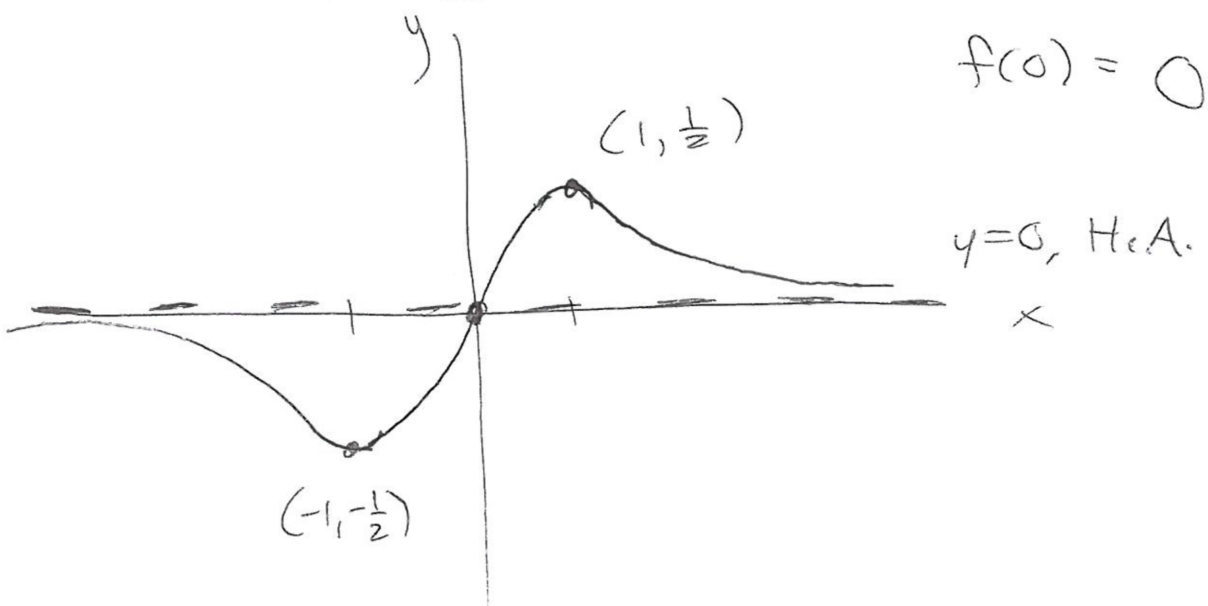
Intervals	$f'' = \frac{2x(x^2-3)}{(x^2+1)^3}$	f
$(-\infty, -\sqrt{3})$	$f''(-10) < 0$	cc dn
$(-\sqrt{3}, 0)$	$f''(-\sqrt{2}) > 0$	cc up
$(0, \sqrt{3})$	$f''(\sqrt{2}) < 0$	cc dn
$(\sqrt{3}, \infty)$	$f''(10) > 0$	cc up

g) $x=-\sqrt{3}$, $x=0$, $x=\sqrt{3}$ inflection pts

h) $f(-1) = \dfrac{-1}{(-1)^2+1} = -\dfrac{1}{2}$ local min

$f(1) = \dfrac{1}{2}$ local max

$f(0) = 0$



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$\boxed{\text{Ex } 5}$ my ex

$f(x) = \frac{x^2-2x-10}{x+5}$ $f'(x) = \frac{x(x+10)}{(x+5)^2}$ $f'' = \frac{50}{(x+5)^3}$

Same instructions

a) Domain: $x \neq -5$

b) $\boxed{\text{V.A} \quad x=-5}$

$$\left.\begin{aligned} &\lim_{x\to\infty} \frac{x^2-2x-10}{x+5} = \lim_{x\to\infty} \frac{2x-2}{1} = \infty \\ &\lim_{x\to-\infty} \frac{x^2-2x-10}{x+5} = \lim_{x\to-\infty} 2x-2 = -\infty \end{aligned}\right\} \text{No H.A}$$

c) $f' = 0 = \frac{x(x+10)}{(x+5)^2}$

$\boxed{x=0,\ x=-10}$ critical numbers

d)

Intervals	f'	f
$(-\infty, -10)$	$f'(-11) > 0$	inc
$(-10, -5)$	$f'(-6) < 0$	dec
$(-5, 0)$	$f'(-1) < 0$	dec
$(0, \infty)$	$f'(1) > 0$	inc

e) $\boxed{\begin{array}{ll} x=-10 & \text{local max} \\ x=0 & \text{local min} \end{array}}$

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f) $f'' = \dfrac{50}{(x+5)^3}$ never 0

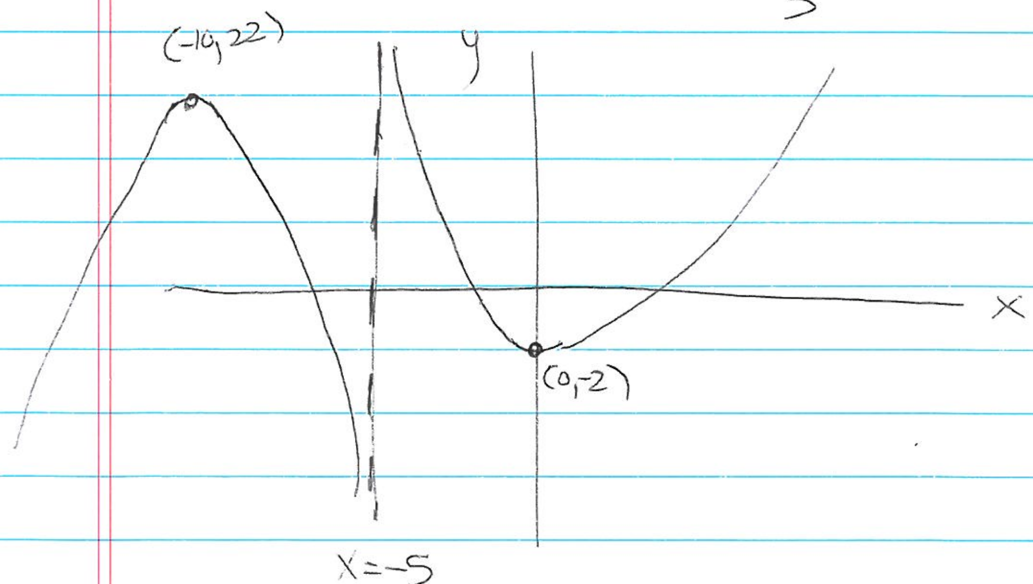
Intervals	$f'' = \frac{50}{(x+5)^3}$	f
$(-\infty, -5)$	$f''(-6) < 0$	ccdn
$(-5, \infty)$	$f''(0) > 0$	cc up

g) no inflection pts ($x=-5$ V.A.)

h) $f(0) = \dfrac{-10}{5} = -2$ local min

$$f(-10) = \frac{(-10)^2 - 2(-10) - 10}{-10+5}$$

$$= \frac{100 + 20 - 10}{5} = \frac{110}{5} = 22$$ local max



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$\boxed{\text{Ex 6 } \text{my example}}$ $y = x\sqrt{x+3}$ $\quad y' = \frac{3(x+2)}{2\sqrt{x+3}}$ $\quad y'' = \frac{3(x+4)}{4(x+3)^{3/2}}$

Find the following
a) domain b) critical numbers c) intervals inc/dec
d) maxs/mins e) intervals cc up/dn f) inflection pts
g) Graph, include labeled max/min, & y-intercept.

a) Domain $x \geq -3$

b) $y' = \frac{3(x+2)}{2\sqrt{x+3}} = 0$

$\boxed{\begin{matrix} x = -2 \\ x = -3 \end{matrix}}$ critical numbers

Note! $x = -2$ is in our domain & $y'(-2) = 0$

$y'(-3) = \text{DNE}$ & -3 is in domain of y

c)

Intervals	$y' = \frac{3(x+2)}{2\sqrt{x+3}}$	y
$(-3, -2)$	$y'(-2.5) < 0$	dec
$(-2, \infty)$	$y'(0) > 0$	inc

d) $x = -2$ local min

e) $x = -4$ is not in our domain

Intervals	$y'' = \frac{3(x+4)}{4(x+3)^{3/2}}$	y
$(-3, \infty)$	$y''(0) > 0$	cc up

f) no inflection pts

g) $y(-2) = -2\sqrt{-2+3} = -2$ $\qquad y(0) = 0$

$\boxed{y(-3) = 0}$

including this since our graph starts at $x = -3$