Finding Inverse Laplace Transforms Using Partial Fractions

If $F(s)$ is a rational function (ratio of polynomials), where $F(s) = \frac{P(s)}{Q(s)}$ then we can use the technique of partial fractions to find $\mathcal{L}^{-1}\{F(s)\}$

Case 1 $Q(s)$ is a product of distinct linear factors, $Q(s) = (s-a_1)(s-a_2)\cdots(s-a_n)$
Then we write $\frac{P(s)}{Q(s)} = \frac{A_1}{s-a_1} + \frac{A_2}{s-a_2} + \cdots + \frac{A_n}{s-a_n}$

We need to find constant values of $A_1, A_2, \ldots, A_n$ so that the equality holds

Examples from *Differential Equations* by Polking, Boggess, and Arnold

$\boxed{\text{Ex1}}$ $F(s) = \frac{1}{(s+2)(s-1)}$

* Step 1: Find the partial fraction decomposition

$$F(s) = \frac{A}{s+2} + \frac{B}{s-1}$$

$$\frac{1}{(s+2)(s-1)} = \frac{A}{s+2} + \frac{B}{s-1}$$

* If we put A & B over a common denominator, we can find the values of A & B that make the equality hold.

$$\frac{1}{(s+2)(s-1)} = \frac{A(s-1) + B(s+2)}{(s+2)(s-1)}$$

Usually we will skip this step and go right to the next one. Basically, since the fractions have the same denominator for them to be equal their numerators must be equal.

So $A(s-1) + B(s+2) = 1$

* Distribute

$$As - A + Bs + 2B = 1$$

* s is a variable, so the coefficients in front of s on the left must equal the Coefficients on the right.

$$\underbrace{As} - A + \underbrace{Bs} + 2B = 1$$

$A + B = 0$ no s's on right

Likewise constants on the left, must equal the constants on the right

$$-A + 2B = 1$$

* Once we have our equations we're ready to solve for A & B

$$A+B=0$$
$$-A+2B=1$$

* Add the equations so that the A's cancel & we get

$$3B=1$$
$$B=1/3 \longrightarrow A=-1/3 \text{ (from 1st equation)}$$

$$F(s)=\frac{-1/3}{s+2}+\frac{1/3}{s-1}$$

$$\mathcal{L}^{-1}\{F(s)\}=\boxed{\frac{-1}{3}e^{-2t}+\frac{1}{3}e^{t}}$$

<u>Case 2</u> Q(s) is a product of linear factors, some of which are repeated

$$\frac{P(s)}{Q(s)}=\frac{P(s)}{(s-a)^n}=\frac{A_1}{(s-a)}+\frac{A_2}{(s-a)^2}+\frac{A_3}{(s-a)^3}+\cdots+\frac{A_n}{(s-a)^n}$$

$\boxed{\text{Ex2}}$ $F(s)=\frac{1}{(s-2)^2(s+1)}$

* Partial Fraction decomposition

$$\frac{1}{(s-2)^2(s+1)}=\frac{A}{s-2}+\frac{B}{(s-2)^2}+\frac{C}{(s+1)}$$

$$\frac{1}{(s-2)^2(s+1)} = \frac{A}{s-2} + \frac{B}{(s-2)^2} + \frac{C}{(s+1)}$$

* Multiply each coefficient by Q(s) divided by their denominator

so A gets multiplied by $\frac{(s-2)^2(s+1)}{(s-2)} = (s-2)(s+1)$

B gets multiplied by $\frac{(s-2)^2(s+1)}{(s-2)^2} = (s+1)$

C gets multiplied by $\frac{(s-2)^2(s+1)}{(s+1)} = (s-2)^2$

$$A(s-2)(s+1) + B(s+1) + C(s-2)^2 = 1$$

$$As^2 - As - 2A + Bs + B + Cs^2 - 4Cs + 4C = 1$$

$A + C = 0$ ← coefficients of s^2 on left & right

$-A + B - 4C = 0$ ← coefficients of s

$-2A + B + 4C = 1$ ← constants on left & right

* Add the 1st 2 equations together to get $B - 3C = 0$

* Multiply the 1st equation by 2 to get $2A + 2C = 0$ & add it to the 3rd to get

$B + 6C = 1$

$$\left.\begin{aligned} B - 3C &= 0 \\ B + 6C &= 1 \end{aligned}\right\} \quad 9C = 1 \rightarrow C = 1/9 \rightarrow B + 6\left(\tfrac{1}{9}\right) = 1$$

$B = \frac{1}{3}$

$A = -1/9$

$$F(s) = \frac{-1/9}{s-2} + \frac{1/3}{(s-2)^2} + \frac{1/9}{(s+1)}$$

$$\mathcal{L}^{-1}\{F(s)\} = -\frac{1}{9}e^{2t} + \frac{1}{3}te^{2t} + \frac{1}{9}e^{-t}$$

Case 3 $Q(s)$ is a product of irreducible quadratics $\frac{P(s)}{Q(s)} = \frac{P(s)}{[(s-\alpha)^2+\beta^2]^n}$

$$= \frac{A_1(s-\alpha) + \beta B_1}{(s-\alpha)^2+\beta^2} + \frac{A_2(s-\alpha) + \beta B_2}{[(s-\alpha)^2+\beta^2]^2} + \dots + \frac{A_n(s-\alpha) + \beta B_n}{[(s-\alpha)^2+\beta^2]^n}$$

$\boxed{\text{Ex 3}}$ $F(s) = \frac{2s^2+9s+11}{(s+1)(s^2+4s+5)} = \frac{A}{s+1} + \frac{B(s-\alpha)+C\beta}{(s-\alpha)^2+\beta^2}$

* step 1! 1st visually determine if our quadratic factors, if it does then it will fall into case 1 or 2. Ours doesn't so we set it equal to $(s-\alpha)^2+\beta^2$ & solve for α & β

$$s^2+\underline{4s}+5 = (s-\alpha)^2+\beta^2$$
$$= s^2\underline{-2\alpha s}+\alpha^2+\beta^2$$

$\alpha = -2$

$5 = \alpha^2+\beta^2$

$\beta = 1$

$$F(s) = \frac{2s^2+9s+11}{(s+1)(s^2+4s+5)} = \frac{A}{s+1} + \frac{B(s+2)+C(1)}{(s+2)^2+1}$$

* Note: I plugged the values of α & β we found back into our decomposition. Now we solve for our coefficients as before.

$$A\left[s^2+4s+5\right] + \left[B(s+2)+C\right](s+1) = 2s^2+9s+11$$

$(s+2)^2+1$

$$As^2 + 4As + 5A + Bs^2 + Bs + 2Bs + 2B + Cs + C = 2s^2 + 9s + 11$$

$A + B = 2$

$4A + B + 2B + C = 9 \rightarrow 4A + 3B + C = 9$ subtract

$5A + 2B + C = 11$

$-A + B = -2$

add

$A + B = 2$

$2B = 0 \quad B = 0 \rightarrow A = 2 \rightarrow C = 1$

$$F(s) = \frac{2}{s+1} + \frac{0\cancel{(s+2)}+1}{(s+2)^2+1}$$

$$\mathcal{L}^{-1}\{F(s)\} = \boxed{2e^{-t} + e^{-2t}\sin t}$$

$\boxed{\text{Ex 4}}$ $F(s) = \dfrac{7s^2+3s+16}{(s+1)(s^2+4)} = \dfrac{A}{s+1} + \dfrac{Bs+2C}{s^2+4}$

note: already in the right form
$\alpha = 0 \,, \quad \beta = 2$

$A(s^2+4) + (Bs+2C)(s+1) = 7s^2+3s+16$

$\underline{As^2} + 4A + \underline{Bs^2} + \underline{Bs} + \underline{2Cs} + 2C = \underline{7s^2} + \underline{3s} + 16$

$$\left.\begin{array}{l} A+B=7 \\ B+2C=3 \end{array}\right\} \quad A-2C=4$$

$$4A+2C=16 \qquad 4A+2C=16$$

$$5A = 20$$
$$A=4$$
$$B=3$$
$$C=0$$

$F(s) = \dfrac{4}{s+1} + \dfrac{3s + \cancel{2} 0}{s^2+4}$

$\mathcal{L}^{-1}\{F(s)\} = \boxed{4e^{-t} + 3\cos(2t)}$

$\boxed{\text{Ex5}}$ $F(s) = \dfrac{7s^2+20s+53}{(s-1)(s^2+2s+5)} = \dfrac{A}{s-1} + \dfrac{B(s-\alpha)+C\beta}{(s-\alpha)^2+\beta^2}$

$(s-\alpha)^2 + \beta^2 = s^2+2s+5$

$s^2 - 2s\alpha + \alpha^2 + \beta^2 = s^2+2s+5$

$\alpha = -1 \quad \beta = 2$

$F(s) = \dfrac{A}{s-1} + \dfrac{B(s+1)+2C}{(s+1)^2+4}$

$$F(s) = \frac{A}{s-1} + \frac{B(s+1) + 2C}{(s+1)^2 + 4}$$

$$A\,(s^2+2s+5) + [B(s+1)+2C](s-1) = 7s^2+20s+53$$

$$Bs + B + 2C$$

$$\underbrace{As^2} + \underline{2As} + 5A + \underbrace{Bs^2} - \cancel{Bs} + \cancel{Bs} - B + \underline{2Cs} - 2C = \underbrace{7s^2} + \underline{20s} + 53$$

$$A + B = 7$$
$$2A + 2C = 20$$
$$5A - B - 2C = 53$$

$$\left.\begin{aligned} A + B &= 7 \\ 7A - B &= 73 \end{aligned}\right\}$$

$$8A = 80$$
$$A = 10$$
$$B = -3$$
$$C = 0$$

$$F(s) = \frac{10}{s-1} + \frac{-3(s+1) + \cancel{2(0)}}{(s+1)^2 + 4}$$

$$\mathcal{L}^{-1}\{F(s)\} = 10e^{t} - 3e^{-t}\cos 2t$$