

Integral Test

Requirements: $\sum_{n=1}^{\infty} a_n$ $a_n = f(n)$ where

f is continuous, positive and decreasing
 $[1, \infty)$

→ If $\int_1^{\infty} f(x) dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent

→ If $\int_1^{\infty} f(x) dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent

Examples from Calculus: The Language of Change by Cohen & Heule

Use the Integral Test to determine whether the series converges or diverges

EX 1 $\sum_{n=1}^{\infty} \frac{1}{n^3}$

* This is a p-series with $p=3 > 1$ so we know this converges; however, they want us to show this with the Integral test. Had they not specified integral test, we should have just used p-series.

We look at $\int_1^{\infty} \frac{1}{x^3} dx$ Basically we just replace the n's with x's

$$\begin{aligned}\int_1^{\infty} \frac{1}{x^3} dx &= \lim_{n \rightarrow \infty} \int_1^n x^{-3} dx = \lim_{n \rightarrow \infty} \frac{x^{-3+1}}{-3+1} \Big|_1^n \\ &= \lim_{n \rightarrow \infty} \frac{x^{-2}}{-2} \Big|_1^n = \lim_{n \rightarrow \infty} -\frac{1}{2} (n^{-2} - 1^{-2}) \\ &= \lim_{n \rightarrow \infty} -\frac{1}{2} \left(\frac{1}{n^2} - 1 \right) = -\frac{1}{2} (0 - 1) \\ &= \boxed{\frac{1}{2}} \text{ converges}\end{aligned}$$

Since the improper integral converges, so does $\sum_{n=1}^{\infty} \frac{1}{n^3}$

Ex 2 $\sum_{n=2}^{\infty} \frac{1}{n \ln(n)}$

* Besides that they told us to do Integral test, we can see that a_n is positive & decreasing but most importantly we can see that we can integrate it

$$\int_2^{\infty} \frac{1}{x \ln x} dx$$

start the
where
series starts

To integrate this improper integral we need a u-substitution. To make our lives easier, 1st we'll do the u-sub & then we'll use the limit.

$$u = \ln x$$

$$du = \frac{1}{x} dx$$

$$u(2) = \ln 2$$

$$u("∞") = \ln "∞" = ∞$$

we can't plug in $∞$ since it is the concept of increasing without bound, but we can see what happens if x increases without bound.

$$\int_2^∞ \frac{1}{x \ln x} dx = \int_{\ln 2}^∞ \frac{1}{u} du = \lim_{n \rightarrow \infty} \int_{\ln 2}^n \frac{1}{u} du$$

$$= \lim_{n \rightarrow \infty} \ln u \Big|_{\ln 2}^n$$

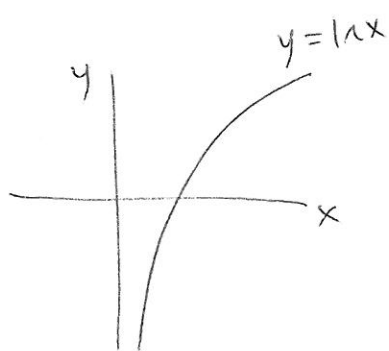
$$= \lim_{n \rightarrow \infty} \ln(n) - \ln(\ln 2)$$

looks weird, but it is just a number

We know $\lim_{n \rightarrow \infty} \ln(n) \rightarrow \infty$

$= \infty$ diverges

So $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges as well



Ex 3

$$\sum_{n=1}^{\infty} n e^{-n^2}$$

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$\int_1^{\infty} x e^{-x^2} dx$ Another u-sub problem:

$$u = -x^2$$

$$du = -2x dx$$

$$-\frac{1}{2} du = x dx$$

$$u(1) = -(1)^2 = -1$$

$$u(\infty) = -(\infty)^2 = -\infty$$

$$\int_1^{\infty} x e^{-x^2} dx = \int_{-1}^{-\infty} -\frac{1}{2} e^u du = \int_{-\infty}^{-1} \frac{1}{2} e^u du$$

$$\left(\int_a^b f(x) dx = -\int_b^a f(x) dx \right)$$

$$= \lim_{n \rightarrow -\infty} \int_n^{-1} \frac{1}{2} e^u du = \lim_{n \rightarrow -\infty} \left. \frac{1}{2} e^u \right|_n^{-1}$$

$$\lim_{n \rightarrow -\infty} \frac{1}{2} e^{-1} - \frac{1}{2} e^n = \frac{1}{2} e^{-1} \quad \text{converges}$$

$$\left(\lim_{n \rightarrow -\infty} e^n = \lim_{n \rightarrow \infty} \frac{1}{e^n} = 0 \right)$$

So $\sum_{n=1}^{\infty} n e^{-n^2}$ converges as well

* Note: $\sum_{n=1}^{\infty} n e^{-n^2}$ does NOT converge to $\frac{1}{2} e^{-1}$, we only know it converges