Integrating with Tables

When we're permitted, integrating with tables can be a real time saver.

In general, identify the form of the integral (ex trig, inverse trig, $\sqrt{u^2-a^2}$, etc) Then do a u-sub to get it to match an integral in a table.

The following examples are from *Calculus* 6e by Edwards & Penney

Ex 1 $\int x\sqrt{4-x^4}\,dx$

* This resembles $\sqrt{a^2-u^2}$ except we have an x^4

Rewriting $\int \underline{x}\sqrt{4-(x^2)^2}\,\underline{dx}$

Let $u=x^2$ $du=2x\,dx$ we have the $x\,dx$ so we divide by 2 $\frac{1}{2}du=x\,dx$

$\int \frac{1}{2}\sqrt{4-u^2}\,du$ * After our u-sub this matches #30 on reference page 7

$$\int\sqrt{a^2-u^2}\,du=\frac{u}{2}\sqrt{a^2-u^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{u}{a}\right)+C$$

so $\int\frac{1}{2}\sqrt{4-u^2}\,du=\boxed{\frac{1}{2}\left[\frac{x^2}{2}\sqrt{4-x^4}+\frac{4}{2}\sin^{-1}\left(\frac{x^4}{2}\right)\right]+C}$ our answer

$\boxed{\text{Ex 2}}$ $\displaystyle\int \frac{x^2}{\sqrt{16x^2+9}}\,dx$

* Again, 1st classify the form. This looks like $\sqrt{u^2+a^2}$

$$\int \frac{x^2\,dx}{\sqrt{(4x)^2+9}}$$

$u=4x$
$du=4\,dx$
$\frac{1}{4}du=dx$

$$=\int \frac{1}{4}\frac{x^2}{\sqrt{u^2+9}}\,du$$

we can't leave this
we know $u=4x \rightarrow x=\frac{u}{4} \rightarrow x^2=\frac{u^2}{16}$

$$=\int \frac{1}{4}\frac{\frac{u^2}{16}}{\sqrt{u^2+9}}\,du = \frac{1}{64}\int \frac{u^2}{\sqrt{u^2+9}}\,du$$

*Now it matches #26 on RP 6

$\frac{1}{4}\cdot\frac{1}{16}$

$$\int \frac{u^2\,du}{\sqrt{a^2+u^2}} = \frac{u}{2}\sqrt{a^2+u^2} - \frac{a^2}{2}\ln\left(u+\sqrt{a^2+u^2}\right)+C$$

$$\frac{1}{64}\int \frac{u^2}{\sqrt{u^2+9}}\,du = \boxed{\frac{1}{64}\left[\frac{(4x)}{2}\sqrt{9+(4x)^2} - \frac{9}{2}\ln\left((4x)+\sqrt{9+(4x)^2}\right)\right]+C}$$

From Stewart Calculus

$\boxed{\text{Ex 3}}$ $\int \sin^{-1}\sqrt{x}\,dx$

* The form is definitely inverse trig, but the $\sqrt{x}$ is a problem. I would like this to be $\sin^{-1}u$. Let's try $u=\sqrt{x}=x^{1/2}$

$$du=\frac{1}{2}x^{-1/2}dx$$
$$=\frac{1}{2\sqrt{x}}dx$$
$$2\sqrt{x}\,du=dx$$

$$\int \sin^{-1}\sqrt{x}\,dx=\int 2\underbrace{\sqrt{x}}_{\text{need to change this}}\sin^{-1}u\,du=\int 2u\sin^{-1}u\,du$$

* On RP9 # 90. $\int u\sin^{-1}u\,du=\frac{2u^2-1}{4}\sin^{-1}u+\frac{u\sqrt{1-u^2}}{4}+C$

$$\int 2u\sin^{-1}u\,du=2\left[\frac{2(\sqrt{x})^2-1}{4}\sin^{-1}\sqrt{x}+\frac{\sqrt{x}\sqrt{1-(\sqrt{x})^2}}{4}\right]+C$$

$$=\boxed{2\left[\frac{x-1}{2}\sin^{-1}\sqrt{x}+\frac{\sqrt{x}\sqrt{1-x}}{4}\right]+C}$$

$\boxed{\text{Ex 4}}$ $\int \frac{\sqrt{2y^2-3}}{y^2}\,dy$

* Looks like $\sqrt{u^2-a^2}$

$= \int \frac{\sqrt{(\sqrt{2}y)^2-3}}{y^2}\,dy$

$u=\sqrt{2}\,y$
$du=\sqrt{2}\,dy$
$\frac{1}{\sqrt{2}}du=dy$

$= \int \frac{\sqrt{u^2-3}}{y^2}\frac{1}{\sqrt{2}}du$

need to get this in terms of u

we know $u=\sqrt{2}\,y \rightarrow \frac{u}{\sqrt{2}}=y$

$\rightarrow y^2=\frac{u^2}{2}$

$= \int \frac{\sqrt{u^2-3}}{\frac{u^2}{2}}\frac{1}{\sqrt{2}}du = \int \frac{2}{\sqrt{2}}\,\frac{\sqrt{u^2-3}}{u^2}\,du$

$\frac{2}{\sqrt{2}}\cdot 1 = \frac{2}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}} = \frac{\cancel{2}\sqrt{2}}{\cancel{(\sqrt{2})^2}} = \sqrt{2}$

$= \int \sqrt{2}\,\frac{\sqrt{u^2-3}}{u^2}\,du$

* Matches #42 on RP 7 $\int \frac{\sqrt{u^2-a^2}}{u^2}du = -\frac{\sqrt{u^2-a^2}}{u} + \ln\left|u+\sqrt{u^2-a^2}\right| + C$

$\int \sqrt{2}\,\frac{\sqrt{u^2-3}}{u^2}du = \sqrt{2}\left[-\frac{\sqrt{2y^2-3}}{(\sqrt{2}y)} + \ln\left|(\sqrt{2}\,y)+\sqrt{(\sqrt{2}y)^2-3}\right|\right] + C$

$= \boxed{-\frac{\sqrt{2y^2-3}}{y} + \sqrt{2}\ln\left|\sqrt{2}\,y+\sqrt{2y^2-3}\right| + C}$

$\boxed{\text{Ex5}}$ $\int y\sqrt{6+4y-4y^2}\,dy$

* Of the form $\sqrt{a^2-u^2}$, but 1st we need to complete the square.

$$6+4y-4y^2 = a^2-(cy+b)^2$$
$$= a^2-c^2y^2-2bcy-b^2$$

$$C=2$$

$$4=-2b(2) \rightarrow b=-1$$
$$6=a^2-(-1)^2$$
$$7=a^2$$

$$\int y\sqrt{7-(2y-1)^2}\,dy \qquad u=2y-1$$
$$du=2dy$$
$$\tfrac{1}{2}du=dy$$

$$=\int \tfrac{1}{2}y\sqrt{7-u^2}\,du$$

need to put in terms of u

$$u=2y-1 \rightarrow y=\frac{u+1}{2}$$

$$=\int \frac{1}{2}\frac{(u+1)}{2}\sqrt{7-u^2}\,du \quad \text{* Split it up}$$

$$=\int \frac{1}{4}u\sqrt{7-u^2}\,du + \int \frac{1}{4}\cdot 1\sqrt{7-u^2}\,du$$

w-sub

$$w=7-u^2$$
$$dw=-2u\,du$$
$$-\tfrac{1}{2}dw=u\,du$$

RP 7 #30

$$\sqrt{a^2-u^2}\,du = \frac{u}{2}\sqrt{a^2-u^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{u}{a}\right)+C$$

$$\int -\frac{1}{8}\overset{w^{1/2}}{\sqrt{w}}\,dw + \frac{1}{4}\left[\frac{(2y-1)}{2}\sqrt{7-(2y-1)^2} + \frac{7}{2}\sin^{-1}\left(\frac{2y-1}{\sqrt{7}}\right)\right] + C$$

$$-\frac{1}{8}\,\frac{2}{3}\,w^{3/2} + \left[\frac{(2y-1)}{8}\sqrt{6+4y-4y^2} + \frac{7}{8}\sin^{-1}\left(\frac{2y-1}{\sqrt{7}}\right)\right] + C$$

$$-\frac{1}{12}\left(7-u^2\right)^{3/2} + \quad ''$$

$$\boxed{-\frac{1}{12}\left(7-(2y-1)^2\right)^{3/2} + \left[\frac{(2y-1)}{8}\sqrt{6+4y-4y^2} + \frac{7}{8}\sin^{-1}\left(\frac{2y-1}{\sqrt{7}}\right)\right] + C}$$