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L'Hospital's Rule

Suppose f and g are differentiable and $g'(x) \neq 0$ near a (except possibly at a)

Suppose that

$\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$

OR

$\lim_{x \to a} f(x) = \pm\infty$ and $\lim_{x \to a} g(x) = \pm\infty$

Basically $\lim_{x \to a} \frac{f(x)}{g(x)}$ is of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$

then

$$\boxed{\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}}$$

* Make sure you check to see if L'Hospital's Rule actually applies before using it!

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Examples from Calculus Early Transcendentals

by Larson, Hostetler, & Edwards

Ex 1 $\lim_{x \to 0} \frac{\sin(4x)}{2x}$ * 1st plug in $\frac{\sin 0}{0} = \text{"}\frac{0}{0}\text{"}$

L'Hospital's Rule applies

$= \lim_{x \to 0} \frac{\cos(4x) \cdot 4}{2}$

* Plug in again

$= \frac{\cos(0) \cdot 4}{2} = \frac{1 \cdot 4}{2} = \frac{4}{2} = \boxed{2}$

Recall!

$\frac{d}{dx}\left(\sin(f(x))\right) =$

$\cos(f(x)) \cdot f'(x)$

from the chain rule.

Ex 2 $\lim_{x \to \infty} \frac{2x+1}{4x^2+x}$ $\text{"}\frac{\infty}{\infty}\text{"}$ L'Hospital's Rule applies

$= \lim_{x \to \infty} \frac{2}{8x+1} = \boxed{0}$

denominator goes to ∞ while the top stay fixed at 2

Ex 3 $\lim_{x \to 1} \frac{\ln x}{x^2-1}$ Plug in $\frac{\ln 1}{1^2-1} = \frac{0}{0}$

$= \lim_{x \to 1} \frac{\frac{1}{x}}{2x} = \frac{\frac{1}{1}}{2 \cdot 1} = \boxed{\frac{1}{2}}$

Ex 4 $\lim_{x\to 0} \dfrac{\sin(x)-x^3}{x-1} = \dfrac{\sin 0 - 0^3}{0-1} = \dfrac{0}{-1} = \boxed{0}$

* If we hadn't checked to see if L'Hospital's rule applied we would have gotten the wrong answer.

Note $\lim_{x\to 0} \dfrac{\sin x - x^3}{x-1} \neq \lim_{x\to 0} \dfrac{\cos x - 3x^2}{1} = 1$

Ex 5 $\lim_{x\to 0^+} \dfrac{e^x-(1+x)}{x^3}$ $\quad \dfrac{e^0-(1+0)}{0^3} = \dfrac{1-1}{0^3} = \dfrac{0}{0}$

$= \lim_{x\to 0^+} \dfrac{e^x - 1}{3x^2}$

$\frac{d}{dx}\left(e^x-(1+x)\right) = \frac{d}{dx}\left(e^x-1-x\right)$
$= e^x - 1$

Plug in $\dfrac{e^0-1}{6x} = \dfrac{0}{0}$

* Sometimes you need to do L'Hospital's rule more than once. Make sure the problem is getting simpler.

$= \lim_{x\to 0^+} \dfrac{e^x}{6x}$ $\quad \dfrac{e^0}{0} = \dfrac{1}{0}$ can't do L'Hospital's Rule again

$= \boxed{\infty}$

Remember that $x \to 0^+$ means that we have positive values of x that are really close to 0, like $\frac{1}{10000}$

$$\frac{e^{1/10000}}{6\left(\frac{1}{10000}\right)} \approx \frac{e^0}{\frac{6}{10000}} = \frac{1}{\frac{6}{10000}} = \frac{10000}{6} \text{ big}$$

Ex 6

$$\lim_{x\to 3} \frac{x-3}{\ln(2x-5)}$$

* Plug in $\frac{0}{\ln(6-5)} = \frac{0}{\ln 1} = \frac{0}{0}$

You should know this

$$= \lim_{x\to 3} \frac{1}{\frac{2}{2x-5}}$$

Recall:

$$\frac{d}{dx}\left(\ln(f(x))\right) = \left(\frac{1}{f(x)}\right)\cdot f'(x)$$

$$= \frac{f'(x)}{f(x)}$$ from the chain rule

$$= \lim_{x\to 3} \frac{2x-5}{2} = \frac{6-5}{2} = \boxed{\frac{1}{2}}$$

Indeterminate Products

Besides $\frac{0}{0}$ or $\frac{\infty}{\infty}$ there are other indeterminate forms.

An indeterminate product happens when $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \infty$ and we want to find $\lim_{x\to a} f(x)g(x)$ "$0\cdot\infty$"

Indeterminate means that we have more work to do to find what it is.

Technique: Rewrite as $\lim_{x\to a} \frac{g(x)}{\frac{1}{f(x)}}$ "$\frac{\infty}{\infty}$"

Or rewrite $\lim_{x\to a} f(x)g(x)$ as $\lim_{x\to a} \frac{f(x)}{\frac{1}{g(x)}}$ "$\frac{0}{0}$"

& then do L'Hospital's Rule.

Recommendation: Leave any $\ln$'s in the numerator!

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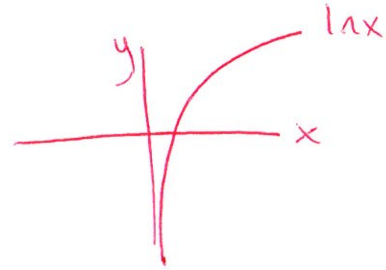
Ex 7 $\lim_{x\to 0^+} (-x\ln x)$ "$0\cdot(-\infty)$"

* Remember



$\lim_{x\to 0^+} \ln x = -\infty$

$\lim_{x\to \infty} \ln x = \infty$

1st we rewrite $\lim_{x\to 0^+} (-x\ln x) = \lim_{x\to 0^+} \dfrac{-\ln x}{\frac{1}{x}}$

but now we have "$\dfrac{-\infty}{\infty}$" so we can use L'Hospital's rule.

$\lim_{x\to 0^+} \dfrac{-\ln x}{x^{-1}} = \lim_{x\to 0^+} \dfrac{-\frac{1}{x}}{-x^{-2}}$ Simplify before plugging in

rewriting so finding the derivative is easier

$= \lim_{x\to 0^+} \dfrac{-\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} \dfrac{-1}{x}\cdot\dfrac{x^2}{-1} = \lim_{x\to 0^+} \dfrac{x^2}{x}$

$= \lim_{x\to 0^+} x = \boxed{0}$

simplifying could be done w/ less steps

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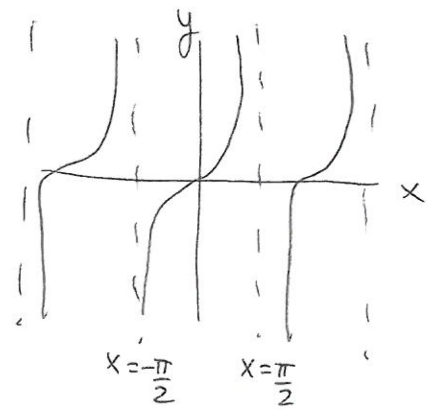
Ex 8 $\lim_{x\to\infty} x\tan\left(\frac{1}{x}\right)$ "∞" $\tan 0$ " = "$\infty \cdot 0$"

$\tan 0 = \frac{\sin 0}{\cos 0} = \frac{0}{1} = 0$

Recall:



$y = \tan x$

$$\lim_{x\to\infty} \frac{1}{x} = 0$$

* Put the less complicated one in the denominator

$$\lim_{x\to\infty} \frac{\tan\left(\frac{1}{x}\right)}{\frac{1}{x}}$$ "$\frac{0}{0}$" time for L'Hospital's Rule

$$= \lim_{x\to\infty} \frac{\tan(x^{-1})}{x^{-1}}$$ optional to make taking derivatives easier

$$= \lim_{x\to\infty} \frac{\sec^2(x^{-1})\cancel{(-x^{-2})}}{\cancel{-x^{-2}}}$$

Recall! $\frac{d}{dx}(\tan(f(x))) = \sec^2(f(x)) \cdot f'(x)$ using the Chain rule.

$$= \lim_{x\to\infty} \sec^2\left(\frac{1}{x}\right)$$

$$= \sec^2 0 = \frac{1}{\cos^2 0} = \frac{1}{1} = \boxed{1}$$

Indeterminate Differences

Another indeterminate form of the type "$\infty - \infty$"

Basically we manipulate it until we can do L'Hospital's rule.

$\boxed{\text{Ex 9}}$ $\lim_{x \to 2^+} \left(\frac{8}{x^2-4} - \frac{x}{x-2} \right)$

$\frac{8}{0} - \frac{2}{0}$

"$\infty - \infty$"

$x > 2$

to figure out $+/- \infty$ just plug in values

$\frac{2.1}{2.1-2} = \frac{2.1}{.1} > 0$

Common denominator

$$= \lim_{x \to 2^+} \left(\underbrace{\frac{8}{(x+2)(x-2)}}_{\text{factored}} - \frac{x(x+2)}{(x+2)(x-2)} \right)$$

$$= \lim_{x \to 2^+} \left(\frac{8 - x(x+2)}{(x+2)(x-2)} \right) = \lim_{x \to 2^+} \frac{8 - x^2 - 2x}{x^2 - 4} \quad \text{"}\frac{0}{0}\text{"}$$

L'Hospital's Rule!

$$= \lim_{x \to 2^+} \frac{-2x-2}{2x} = \frac{-4-2}{4} = \frac{-6}{4} = \boxed{\frac{-3}{2}}$$

Ex 10 $\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{x^2}\right)$ "$\infty-\infty$"

$$=\lim_{x\to 0^+}\left(\frac{x}{x^2}-\frac{1}{x^2}\right)$$

$$=\lim_{x\to 0^+}\left(\frac{x-1}{x^2}\right)$$

* L'Hospital's rule doesn't apply $\frac{-1}{0}$

To determine sign plug in values close to 0 since $x\to 0^+$ $x>0$

$$\left(\frac{.01-1}{(.01)^2}<0\right)$$

so $\lim_{x\to 0^+}\left(\frac{x-1}{x^2}\right)=\boxed{-\infty}$

Ex 11 $\lim_{x\to\infty}\left(\sqrt{x^2+5x+2}-x\right)$ "$\infty-\infty$"

Conjugates

$$=\lim_{x\to\infty}\left(\sqrt{x^2+5x+2}-x\right)\frac{\left(\sqrt{x^2+5x+2}+x\right)}{\sqrt{x^2+5x+2}+x}$$

$$=\lim_{x\to\infty}\frac{(x^2+5x+2)-x^2}{\sqrt{x^2+5x+2}+x}$$

$$=\lim_{x\to\infty}\frac{5x+2}{\sqrt{x^2+5x+2}+x}$$ "$\frac{\infty}{\infty}$"

* Although technically we can do L'Hospital's Rule our derivatives will be a mess

Instead of L'Hospital's rule let's use our previous technique

$$\lim_{x\to\infty} \frac{5x+2}{\sqrt{x^2+5x+2}+x} \cdot \frac{\frac{1}{x}}{\frac{1}{x}}$$

Biggest power of x in denominater

$$= \lim_{x\to\infty} \frac{5+\frac{2}{x}}{\frac{1}{x}\sqrt{x^2+5x+2}+1}$$

$$= \lim_{x\to\infty} \frac{5+\frac{2}{x}}{\sqrt{\frac{x^2}{x^2}+\frac{5x}{x^2}+\frac{2}{x^2}}+1}$$

$$\frac{1}{x}\sqrt{x^2+5x+2}$$

$$= \sqrt{\frac{1}{x^2}}\sqrt{x^2+5x+2}$$

$$= \sqrt{\frac{1}{x^2}(x^2+5x+2)}$$

$$= \lim_{x\to\infty} \frac{5+\frac{2}{x}}{\sqrt{1+\frac{5}{x}+\frac{2}{x^2}}+1}$$

$$= \frac{5+0}{\sqrt{1}+1}$$

$$= \boxed{\frac{5}{2}}$$

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Indeterminate Powers

$\lim_{x\to a}[f(x)]^{g(x)}$ when we get the types

"1^∞" "∞^0" or "0^0"

TRICKY! "∞^∞" $\to \infty$, "0^∞" $\to 0$ Less tricky "∞^1" $\to \infty$

Technique: ① Set $L = \lim_{x\to a}[f(x)]^{g(x)}$

② $\ln L = \lim_{x\to a} \ln(f(x))^{g(x)}$

$= \underbrace{\lim_{x\to a} g(x)\ln(f(x)) = \ell}_{\text{find this using technique on previous pages}}$

so $\ln L = \ell$

or ③ $\boxed{L = e^{\ell}}$ answer

$\boxed{\text{Ex 12}}$ $\lim_{x\to\infty} x^{1/x}$ "∞^0"

Step 1: $L = \lim_{x\to\infty} x^{\frac{1}{x}}$

Step 2: $\ln L = \lim_{x\to\infty} \ln x^{1/x}$

$= \lim_{x\to\infty} \frac{1}{x}\ln x$ "$0\cdot\infty$"

$= \lim_{x\to\infty} \frac{\ln x}{x}$ } You can't do $\frac{\ln \cancel{x}}{\cancel{x}} = \ln$ Don't even think about it :)

"$\frac{\infty}{\infty}$"

$= \lim_{x\to\infty} \frac{1/x}{1} = 0$ so $\ln L = 0 \longrightarrow L = \boxed{1}$ answer

$L = e^0 = 1$

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Ex 13 $\lim_{x\to 0^+} (1+8x)^{\frac{1}{x}}$ "1^{∞}"

$L = \lim_{x\to 0^+} (1+8x)^{\frac{1}{x}}$

$\ln L = \lim_{x\to 0^+} \frac{1}{x}\ln(1+8x)$ "$\infty \ln 1$" = "$\infty \cdot 0$"

$= \lim_{x\to 0^+} \frac{\ln(1+8x)}{x}$ "$\frac{\ln 1}{0}$" = "$\frac{0}{0}$"

$= \lim_{x\to 0^+} \frac{\frac{8}{1+8x}}{1}$

$= \frac{8}{1+0} = 8$

$\ln L = 8$

$L = \boxed{e^8}$

Ex 14 $\lim_{x\to 0^+} (e^x + x)^{1/x}$ "1^{∞}"

$L = \lim_{x\to 0^+} (e^x + x)^{1/x}$

$\ln L = \lim_{x\to 0^+} \frac{1}{x}\ln(e^x + x)$ "$\infty \ln(e^0)$" = "$\infty \cdot 0$"

$= \lim_{x\to 0^+} \frac{\ln(e^x+x)}{x}$ "$\frac{0}{0}$"

$= \lim_{x\to 0^+} \frac{\frac{1}{e^x+x}(e^x+1)}{1}$

$= \lim_{x\to 0^+} \frac{e^x+1}{e^x+x} = \frac{e^0+1}{e^0+0} = \frac{2}{1} = 2$

$\ln L = 2$

$L = \boxed{e^2}$

Ex 15 $\lim_{x\to 1^+} (x-1)^{\ln x}$ "$0^{\ln 1}$" = "0^0"

$$L = \lim_{x\to 1^+} (x-1)^{\ln x}$$

$$\ln L = \lim_{x\to 1^+} \ln\left((x-1)^{\ln x}\right)$$

$$= \lim_{x\to 1^+} \ln x \ln(x-1) \qquad \text{"}\ln 1\cdot(-\infty)\text{"} = \text{"}0\cdot(-\infty)\text{"}$$

* this is exciting, we have to put one of the ln's in the denominator

$$\lim_{x\to 1^+} \frac{\ln(x-1)}{\frac{1}{\ln x}} \qquad \frac{\text{"}-\infty\text{"}}{\text{"}\infty\text{"}}$$

$$= \lim_{x\to 1^+} \frac{\ln(x-1)}{(\ln x)^{-1}}$$

getting ready for the derivative

$$= \lim_{x\to 1^+} \frac{\frac{1}{x-1}\cdot 1}{-(\ln x)^{-2}\cdot\frac{1}{x}}$$

Chain rule

$$\underbrace{= \lim_{x\to 1^+} \frac{\frac{1}{x-1}}{\frac{-1}{(\ln x)^2 x}} = \lim_{x\to 1^+} \frac{-(\ln x)^2 x}{x-1}}_{\text{simplifying}} \qquad \text{"}\frac{0}{0}\text{"}$$

L'Hospital's rule... again

$$= \lim_{x\to 1^+} \frac{-\left(2(\ln x)\frac{1}{x}\right)x - \left(-(\ln x)^2\right)\cdot 1}{1} = \frac{0}{1} = 0$$

so $\ln L = 0$ $\quad L = \boxed{1}$

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$\boxed{\text{Ex } 16}$ $\lim_{x \to 0^+} (\sin x)^x$ "0^0"

$L = \lim_{x \to 0^+} (\sin x)^x$

$\ln L = \lim_{x \to 0^+} x \ln(\sin x)$ "$0 \ln(\sin 0)$" $=$ "$0(-\infty)$"

$= \lim_{x \to 0^+} \dfrac{\ln(\sin x)}{\frac{1}{x}}$

$= \lim_{x \to 0^+} \dfrac{\ln(\sin x)}{x^{-1}}$

$= \lim_{x \to 0^+} \dfrac{\frac{1}{\sin x} \cdot \cos x}{-x^{-2}}$

$= \lim_{x \to 0^+} \dfrac{\frac{\cos x}{\sin x}}{-\frac{1}{x^2}}$

$= \lim_{x \to 0^+} -\dfrac{x^2 \cos x}{\sin x}$ "$\dfrac{0^2 \cos 0}{\sin 0}$" $=$ "$\dfrac{0}{0}$"

$= \lim_{x \to 0^+} \dfrac{-2x \cos x - x^2(-\sin x)}{\cos x}$ } Product rule

$= \dfrac{-2 \cdot 0 \cos 0 - 0^2(-\sin 0)}{\cos 0} = \dfrac{0}{1} = 0$

$\ln L = 0$

$L = e^0 = \boxed{1}$