

Limit Comparison Test

Suppose $a_n > 0, b_n > 0$ and $\lim_{n \rightarrow \infty} \left(\frac{a_n}{b_n}\right) = L$ where L is finite and $L > 0$, then the 2 series $\sum a_n$ and $\sum b_n$ both converge or both diverge

* I will be re-doing problems from the Comparison Test WS with the Limit Comparison Test when applicable

Ex 1

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1}$$

* n^0 degree of top = n^{-2} degree of bottom = $\frac{1}{n^2}$

$$\lim_{n \rightarrow \infty} \left(\frac{\frac{1}{n^2+1}}{\frac{1}{n^2}} \right) = \lim_{n \rightarrow \infty} \frac{n^2}{n^2+1} = 1 > 0$$

Both converge or diverge by the Limit Comparison Test. $\sum_{n=1}^{\infty} \frac{1}{n^2}$ p-series, $p=2 > 1$ converges

so $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges

Ex 2

$$\sum_{n=1}^{\infty} \frac{1}{3n^2+2}$$

$$\lim_{n \rightarrow \infty} \frac{\frac{1}{3n^2+2}}{\frac{1}{n^2}} = \lim_{n \rightarrow \infty} \frac{n^2}{3n^2+2} = \frac{1}{3} > 0$$

since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ p-series, $p=2 > 1$ converges

$\sum_{n=1}^{\infty} \frac{1}{3n^2+2}$ converges by the Limit Comparison test

Ex 3

$$\sum_{n=3}^{\infty} \frac{1}{n-2}$$

$$\lim_{n \rightarrow \infty} \frac{\frac{1}{n-2}}{\frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{n}{n-2} = 1 > 0$$

since $\sum_{n=3}^{\infty} \frac{1}{n}$ Harmonic series diverges

$\sum_{n=3}^{\infty} \frac{1}{n-2}$ diverges by the Limit Comparison Test

* This is the minimal work needed to get full credit on this problem using this test *

Ex 4

$$\sum_{n=0}^{\infty} \frac{1}{3^{n+1}}$$

$$\lim_{n \rightarrow \infty} \frac{\frac{1}{3^{n+1}}}{\frac{1}{3^n}} = \lim_{n \rightarrow \infty} \frac{3^n}{3^{n+1}} = \frac{1}{3} > 0$$

$\sum_{n=0}^{\infty} \frac{1}{3^n}$ Geometric series $r = \frac{1}{3}$ $|\frac{1}{3}| < 1$ converges

So $\sum_{n=0}^{\infty} \frac{1}{3^{n+1}}$ converges by the Limit Comparison test

EX 5

$$\sum_{n=0}^{\infty} \frac{2^n}{3^n + 5}$$

$$\lim_{n \rightarrow \infty} \frac{2^n}{3^n + 5} = \lim_{n \rightarrow \infty} \frac{3^n 2^n}{(3^n + 5) 2^n} = \lim_{n \rightarrow \infty} \frac{2^n}{3^n} = 1 > 0$$

$$\sum_{n=0}^{\infty} \frac{2^n}{3^n} = \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n \quad \text{Geometric series } r = \frac{2}{3}$$

$|2/3| < 1$ converges

So $\sum_{n=0}^{\infty} \frac{2^n}{3^n + 5}$ converges by the Limit Comparison test

EX 6

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3 + 1}}$$

degree of top = 0
degree of bottom = $3/2$ ($\sqrt{n^3 + 1} = (n^3 + 1)^{1/2}$)

$$\lim_{n \rightarrow \infty} \left(\frac{\frac{1}{\sqrt{n^3 + 1}}}{\frac{1}{n^{3/2}}} \right) = \lim_{n \rightarrow \infty} \frac{n^{3/2}}{\sqrt{n^3 + 1}} = 1 > 0$$

$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$$

p-series $p = 3/2 > 1$ converges

So

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3 + 1}}$$

converges as well by the LCT

EX 7

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^3 - 1}}$$

$$\lim_{n \rightarrow \infty} \left(\frac{\frac{1}{\sqrt{n^3 - 1}}}{\frac{1}{n^{3/2}}} \right) = \lim_{n \rightarrow \infty} \frac{n^{3/2}}{\sqrt{n^3 - 1}} = 1 > 0$$

$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$$

p-series $p = 3/2 > 1$ converges

So

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^3 - 1}}$$

converges as well by the limit comparison test

EX 8

$$\sum_{n=1}^{\infty} \frac{4^n}{3^n - 1}$$

$$\lim_{n \rightarrow \infty} \frac{4^n}{3^n - 1} = \lim_{n \rightarrow \infty} \frac{4^n 3^n}{(3^n - 1) 4^n} = \infty$$

$\sum_{n=1}^{\infty} \left(\frac{4}{3}\right)^n$ Geometric series $r = \frac{4}{3}$ $|\frac{4}{3}| > 1$
diverges

so $\sum_{n=1}^{\infty} \frac{4^n}{3^n - 1}$ diverges as well by the

Limit Comparison Test

EX 9

$$\sum_{n=1}^{\infty} \frac{1}{3^n + 2}$$

$$\lim_{n \rightarrow \infty} \frac{1}{3^n + 2} = \lim_{n \rightarrow \infty} \frac{3^n}{3^n + 2} = 1 > 0$$

$$\sum_{n=1}^{\infty} \frac{1}{3^n}$$

Geometric series with $r = \frac{1}{3}$
 $|\frac{1}{3}| < 1$ converges

so $\sum_{n=1}^{\infty} \frac{1}{3^n + 2}$ converges by Limit Comparison test