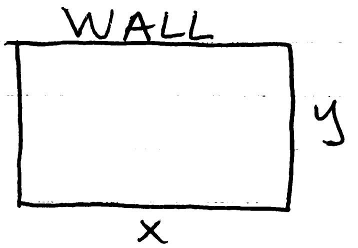
Optimization

Examples from Calculus by Jon Rogawski

Ex 1 A landscape architect wishes to enclose a rectangular garden on one side by a brick wall costing \$30/ft and on 3 sides by a metal fence costing \$10/ft. If the area of the garden is 1,000 ft^2, find the dimensions of the garden that will minimize cost.



$$A = xy = 1000$$

$$\text{Cost} = \$30x + \$10(2y + x)$$

$$C = 30x + 20y + 10x$$
$$= 40x + 20y$$

$$y = 1000/x$$

$$C = 40x + 20000x^{-1}$$

$$C' = 40 + (-20000x^{-2}) = 0$$

$$40 = 20000x^{-2}$$

$$x^2 = \frac{2000\cancel{0}}{4\cancel{0}} = 500$$

$$x = \sqrt{500}$$

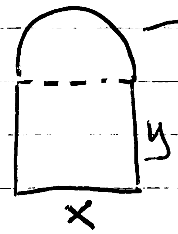
$$C'' = 40000x^{-3}$$

$$C''(\sqrt{500}) > 0 \text{ max}$$

Dimensions: $\boxed{x = \sqrt{500} \quad y = \frac{1000}{\sqrt{500}} = 2\sqrt{500}}$

Note the basic steps! 1. Draw picture
2. Find equations to express the relationships
3. Get the equation to be maximized/minimized in terms of 1 variable
4. Take its derivative and find the critical points
5. Take the 2nd derivative to check for max/min

Ex2 Suppose that 600ft of fencing are used to enclose a corral in the shape of a rectangle with a semicircle whose diameter is a side of the rectangle shown below. Find the dimensions of the corral of maximum area.



$C = \frac{1}{2}(2\pi r)$ we know $D = x = 2r$

circumference of a semicircle

$$P = x + 2y + \frac{1}{2}\left(2\pi\left(\tfrac{1}{2}x\right)\right) = 600$$

$$x + 2y + \pi/2\, x = 600$$

$$(1 + \pi/2)x + 2y = 600 \rightarrow y = \tfrac{1}{2}\left[600 - (1+\tfrac{\pi}{2})x\right]$$

$$A = xy + \tfrac{1}{2}\pi r^2 = xy + \tfrac{1}{2}\pi\left(\tfrac{1}{2}x\right)^2$$

$$A = x\left[\tfrac{1}{2}\left(600 - (1+\pi/2)x\right)\right] + \tfrac{1}{8}\pi x^2$$

$$A = 300x - \frac{(1+\pi/2)}{2}x^2 + \tfrac{1}{8}\pi x^2$$

$$A' = 300 - (1+\pi/2)x + \tfrac{1}{4}\pi x = 0$$

$$300 - x - \pi/4\, x = 0 \qquad 300 = (1 + \pi/4)x$$

$$x = \frac{300}{1+\pi/4} = \frac{300}{\frac{4+\pi}{4}} = \boxed{\frac{1200}{4+\pi}}$$

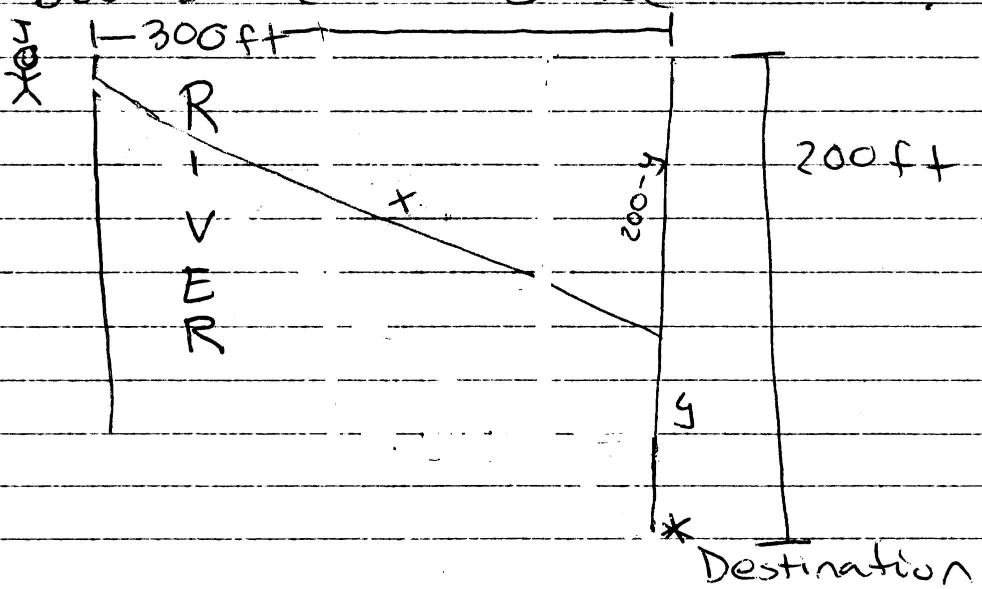
$$y = \frac{1}{2}\left[600 - \left(1+\frac{\pi}{2}\right)x\right]$$

$$= \frac{1}{2}\left[600 - \left(\frac{2+\pi}{2}\right)\left(\frac{1200}{4+\pi}\right)\right] = \frac{1}{2}\left[600 - \frac{(2+\pi)600}{4+\pi}\right] = 300\left[\frac{4+\pi-2-\pi}{4+\pi}\right] = \boxed{\frac{600}{4+\pi}}$$

Check: $A'' = -\pi/4 < 0$ $\quad A''\left(\frac{1200}{4+\pi}\right) < 0$ max

This is a little long

Ex3 Janice can swim 3 mph and run 8 mph. She is standing at one bank of a river that is 300ft wide and wants to reach a point located 200 ft downstream on the other side as quickly as possible. She will swim diagonally across the river and then jog along the river bank. Find the best route for Janice to take.



$$300^2 + (200-y)^2 = x^2$$

Time traveled $= T = \dfrac{x}{3} + \dfrac{y}{8}$ $\qquad \dfrac{\text{Distance}}{\text{Rate}} = t$

$$x = \sqrt{300^2 + (200-y)^2}$$

$$T = \frac{1}{3}\sqrt{300^2 + (200-y)^2} + \left(\frac{1}{8}\right)y$$
$$= \frac{1}{3}(300^2 + (200-y)^2)^{1/2} + y/8$$

$$\frac{dT}{dy} = \frac{1}{6}(300^2 + (200-y)^2)^{-1/2}(-2(200-y)) + \frac{1}{8} = 0$$

$$\left(\frac{1}{8}\right)^2 = \left(\frac{2(200-y)}{6\sqrt{300^2+(200-y)^2}}\right)^2$$

$$\frac{9}{64} = \frac{(200-y)^2}{300^2+(200-y)^2}$$

$$\frac{9}{64}\left(300^2 + (200-y)^2\right) = (200-y)^2$$

$$\frac{9}{64}(300^2) + \frac{9}{64}(200-y)^2 = (200-y)^2$$

$$\frac{9}{64}(300)^2 = \frac{55}{64}(200-y)^2$$

$$\frac{3}{\cancel{8}}(300) = \frac{\sqrt{55}}{\cancel{8}}(200-y)$$

$$\frac{900}{\sqrt{55}} = 200-y$$

$$y = 200 - \frac{900}{\sqrt{55}}$$

Swim to $\frac{900}{\sqrt{55}} \approx 121.4$ ft

downstream the point directly across from the river