Orthogonal Trajectories

An orthogonal trajectory of a family of curves is a curve that intersects each curve of the family at right angles.

Technique:
1. Find $\frac{dy}{dx}$ (slope)
2. Take its negative reciprical ($\perp$)
3. Eliminate k, if necessary
4. Solve differential equation

Find the orthogonal trajectories of the family of curves

Ex 1 $\quad y = ke^x$

Step1 slope $\quad \frac{dy}{dx} = ke^x$

step2 $\perp$ $\quad \frac{dy}{dx} = \frac{-1}{ke^x}$

Step3 k, we're given $y = ke^x$ so $k = \frac{y}{e^x}$

Plug into step 2

$$\frac{dy}{dx} = \frac{-1}{\frac{y}{\cancel{e^x}}\cancel{e^x}} = \frac{-1}{y}$$

Step 4 $\int y\,dy = \int -dx$

$$\boxed{\frac{1}{2}y^2 = -x + C}$$

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Ex 2 $\quad y^3 = kx^2$

Step 1 $\quad 3y^2 \frac{dy}{dx} = 2kx$ using implicit differentiation

$$\frac{dy}{dx} = \frac{2kx}{3y^2}$$

Step 2 $\quad \frac{dy}{dx} = \frac{-3y^2}{2kx}$

Step 3 We know $y^3 = kx^2$ so $k = \frac{y^3}{x^2}$

$$\frac{dy}{dx} = \frac{-3y^2}{2\left(\frac{y^3}{x^2}\right)x} = \frac{-3x}{2y}$$

Step 4 $\quad \int 2y\,dy = \int -3x\,dx$

$$y^2 = -\frac{3}{2}x^2 + C$$

$\boxed{y^2 + \frac{3}{2}x^2 = C}$ Family of ellipses

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Ex 3 $y = e^{kx}$

Step 1 $\frac{dy}{dx} = ke^{kx}$

Step 2 $\frac{dy}{dx} = \frac{-1}{ke^{kx}}$

Step 3 $y = e^{kx} \rightarrow \ln y = kx \quad k = \frac{\ln y}{x}$

$$\frac{dy}{dx} = \frac{-1}{\frac{\ln y}{x} e^{\frac{\ln y}{x} x}}$$

$$= \frac{-1}{\frac{\ln y}{x} e^{\ln y}}$$

$$= \frac{-x}{(\ln y)\, y}$$

Step 4 $\int y \ln y \, dy = \int -x \, dx$

Integration by parts!

LIATE

$u = \ln y \qquad v = \frac{1}{2}y^2$

$du = \frac{1}{y} dy \quad dv = y\,dy$

$uv - \int v\,du$

$\frac{1}{2}y^2 \ln y - \int \frac{1}{2}y^2 \cdot \frac{1}{y} dy = -\frac{1}{2}x^2 + C$

$\frac{1}{2}y^2 \ln y - \int \frac{1}{2} y\,dy = -\frac{1}{2}x^2 + C$

$$\boxed{\frac{1}{2}y^2 \ln y - \frac{1}{4}y^2 = -\frac{1}{2}x^2 + C}$$

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Ex 4 $\quad y=\sqrt{kx} = (kx)^{1/2}$

Step 1 $\quad \frac{dy}{dx} = \frac{1}{2}(kx)^{-1/2} \cdot k = \frac{k}{2\sqrt{kx}}$

step 2 $\quad \frac{dy}{dx} = \frac{-2\sqrt{kx}}{k}$

Step 3 $\quad y=\sqrt{kx} \longrightarrow y^2 = kx \quad k = \frac{y^2}{x}$

$$\frac{dy}{dx} = \frac{-2\sqrt{\frac{y^2}{x}\cdot x}}{\frac{y^2}{x}} = \frac{-2\sqrt{y^2}}{\frac{y^2}{x}} = \frac{-2y}{\frac{y^2}{x}}$$

$$\frac{dy}{dx} = \frac{-2x}{y}$$

step 4

$$\int y\,dy = \int -2x\,dx$$

$$\frac{1}{2}y^2 = -x^2 + C$$

$$\boxed{\frac{1}{2}y^2 + x^2 = C}$$

Family of ellipses