

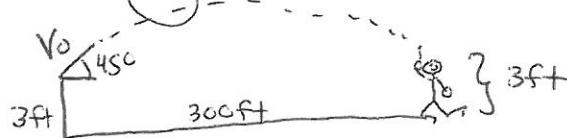
# Projectile Motion

Examples from Calculus by Larson, Hostetler, Edwards

**Ex1** A baseball, hit 3 ft above the ground, leaves the bat at an angle of  $45^\circ$  and is caught by an outfielder 300ft from home plate. What is the initial speed of the ball, and how high does it rise if it is caught 3ft above the ground?

\*We want to be able to solve this problem starting with our acceleration vector & working from there. For these problems we are assuming that there is no air resistance.

$$\vec{a} = \langle 0, -32 \rangle$$



acceleration due to gravity  
if we had meters we'd have  
used  $9.8 \text{ m/s}^2$

$$\vec{v} = \langle 0, -32t \rangle + \vec{c}$$

$$\rightarrow \vec{v}(0) = \langle |\vec{v}(0)| \cos 45^\circ, |\vec{v}(0)| \sin 45^\circ \rangle$$

initial velocity  $|\vec{v}(0)|$  = initial speed, let's call it  $v_0$

Just breaking  $\vec{v}(0)$  down  
into its components in the  
x & y directions

$$\vec{v}(0) = \langle v_0 \frac{1}{\sqrt{2}}, v_0 \frac{1}{\sqrt{2}} \rangle$$

Note:  $\vec{v} = \langle 0, -32t \rangle + \vec{c}$

$$\text{so } \vec{v}(0) = \langle 0, -32 \cdot 0 \rangle + \vec{c} = \vec{c}$$

$$\text{so } \vec{c} = \left\langle \frac{v_0}{\sqrt{2}}, \frac{v_0}{\sqrt{2}} \right\rangle$$

Plugging in:  $\vec{v} = \left\langle \frac{v_0}{\sqrt{2}}, \frac{v_0}{\sqrt{2}} - 32t \right\rangle$

Position:  $\vec{r} = \left\langle \frac{v_0}{\sqrt{2}}t, \frac{v_0}{\sqrt{2}}t - 16t^2 \right\rangle + \vec{d}$

$$\vec{r}(0) = \langle 0, 3 \rangle$$

$\begin{matrix} \text{height of the ball} \\ \text{leaving the bat} \end{matrix}$

$$\rightarrow \vec{d} = \langle 0, 3 \rangle$$

$$\vec{r} = \left\langle \frac{v_0}{\sqrt{2}}t, \frac{v_0}{\sqrt{2}}t - 16t^2 + 3 \right\rangle$$

\* I like to start these problems by drawing a picture & then finding  $\vec{v}$  &  $\vec{r}_0$ . After we have  $\vec{v}$  &  $\vec{r}_0$  we can figure out how to answer the remaining questions.

1st Let's find  $v_0$ . We now know  $\vec{r}$  but we also know when the ball is 300ft away it should be 3ft off the ground.

To find  $v_0$  we'll set the  $x$  component of position equal to 300 ft & the  $y$  comp of position equal to 3ft.

$$\underbrace{\frac{v_0}{\sqrt{2}} t}_{=} = 300$$

$$\underbrace{\frac{v_0}{\sqrt{2}} t - 16t^2 + 3}_{=} = 3$$

2 equations, 2 unknowns : Let's be slightly fancy:

$$> 300 - 16t^2 + 3 = 3$$

$$300 = 16t^2$$

$$t^2 = \frac{300}{16}$$

$$t = \frac{\sqrt{300}}{4} \quad \leftarrow \text{time it takes to catch the ball}$$

$$\frac{v_0}{\sqrt{2}} \left( \frac{\sqrt{300}}{4} \right) = 300$$

\* Just plugging the time into the  $\frac{v_0}{\sqrt{2}} t = 300$  equation

$$\frac{v_0}{4} \sqrt{150} = 300$$

$$v_0 = \frac{1200}{\sqrt{150}} \text{ ft/s} \approx 98 \text{ ft/s}$$

How high does it rise? (This is equivalent to asking for the max height)

The max height occurs when the y component of velocity is 0 (We know from Calc 1 that the slope is 0 at a max or min. The derivative of position is velocity)

$$\frac{V_0}{\sqrt{2}} - 32t = 0$$

$$V_0 = \frac{1200}{\sqrt{150}}$$

$$\frac{1200}{\sqrt{150} \sqrt{2}} - 32t = 0$$

$$t = \frac{1200}{\sqrt{300} \cdot 32} \quad \leftarrow \text{time the max height occurs}$$

Now we go back to position & plug this time into the y comp

$$\frac{V_0}{\sqrt{2}} \left( \frac{1200}{\sqrt{300} \cdot 32} \right) - 16 \left( \frac{1200}{\sqrt{300} \cdot 32} \right)^2 + 3 = \text{max height}$$

$$\frac{1200}{\sqrt{300}} \left( \frac{1200}{\sqrt{300} \cdot 32} \right) - 16 \left( \frac{1200}{\sqrt{300} \cdot 32} \right)^2 + 3 = \text{Max height}$$

$$\left( \frac{1200^2}{300} \right) \left[ \frac{1}{32} - \frac{16}{32^2} \right] + 3 = \left( \frac{1200^2}{300} \right) \left[ \frac{16}{32^2} \right] + 3$$

got tired of simplifying = 78ft

**Ex 2** Determine the maximum height and range of a projectile fired at a height of 1.5 meters above the ground with an initial speed of 100 m/s and at an angle  $30^\circ$  above the horizontal. P5

\* Draw a picture



$$\vec{a} = \langle 0, -9.8 \rangle$$

gravity for meters

$$\vec{v} = \langle 0, -9.8t \rangle + \vec{c}$$

$$\begin{aligned}\vec{r}(0) &= \langle 100\cos 30^\circ, 100\sin 30^\circ \rangle \\ &= \left\langle 100 \frac{\sqrt{3}}{2}, 100 \left(\frac{1}{2}\right) \right\rangle = \langle 50\sqrt{3}, 50 \rangle\end{aligned}$$

$$\vec{v} = \langle 50\sqrt{3}, 50 - 9.8t \rangle$$

$$\vec{r} = \langle 50\sqrt{3}t, 50t - 4.9t^2 \rangle + \vec{d}$$

$$\vec{r}(0) = \langle 0, 1.5 \rangle$$

$$\vec{r} = \langle 50\sqrt{3}t, 50t - 4.9t^2 + 1.5 \rangle$$

\* The range of the projectile is how far it gets in the x direction from where it starts. The y value of position here is 0 since the projectile wasn't caught or didn't hit any objects.

Set y-comp of position equal to 0. P6

$$50t - 4.9t^2 + 1.5 = 0$$

$$t = \frac{-50 \pm \sqrt{50^2 - 4(-4.9)(1.5)}}{2(-4.9)} = -0.0299 \text{ or } 10.234$$

Negative time. This is extending the path of the projectile from before it was fired. Not what we want.

Time it takes projectile to land / time of max range.

\* Plug into x-comp of position

$$50\sqrt{3}(10.234) = \boxed{886.29 \text{ m}}$$

max range

Max height again this is the time when the y-comp of velocity is 0

$$50 - 9.8t = 0 \quad t = \frac{50}{9.8}$$

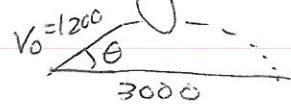
Plug into y-comp of position

$$50\left(\frac{50}{9.8}\right) - 4.9\left(\frac{50}{9.8}\right)^2 + 1.5 = \text{max height}$$

$$\frac{50^2}{9.8} \left(1 - \frac{9.8}{2} \frac{1}{9.8}\right) + 1.5 = \frac{50(25)}{9.8} + 1.5 = \boxed{129 \text{ m}}$$

**Ex3** A shot is fired from a gun with a muzzle speed of 1200 ft/s to a target 3000 ft away. Determine the minimum angle of elevation of the gun.

$$\vec{a} = \langle 0, -32 \rangle$$



$$\vec{v} = \langle 1200 \cos \theta, 1200 \sin \theta - 32t \rangle$$

$$\vec{F} = \langle 1200 \cos \theta t, 1200 \sin \theta t - 16t^2 \rangle + \vec{a}$$

I'm assuming that the gun & target are at the same height.

$$1200 \cos \theta t = 3000 \quad 1200 \sin \theta t - 16t^2 = 0$$

$$t(1200 \sin \theta - 16t) = 0$$

$$t=0 \\ \text{Start position}$$

$$t = \frac{1200 \sin \theta}{16}$$

time it takes to hit target

$$1200 \cos \theta \frac{1200 \sin \theta}{16} = 3000$$

Recall:  $\sin 2\theta = 2 \sin \theta \cos \theta$  so

$$\frac{600}{16} (1200) (2 \cos \theta \sin \theta) = 3000 \quad \text{becomes}$$

$$\frac{600}{16} (12 \cos \theta \sin \theta) = 3000$$

$$\sin 2\theta = \frac{3(16)}{12(60)} = \frac{16}{4(60)} = \frac{4}{60} = \frac{2}{30} = \frac{1}{15}$$

$$2\theta = \sin^{-1}\left(\frac{1}{15}\right)$$

$$\theta = \frac{\sin^{-1}\left(\frac{1}{15}\right)}{2} = \boxed{1.91^\circ}$$