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RELATED RATES

In this topic we show how implicit differentiation and the chain rule can be used to calculate the rate of change of one variable in terms of the rate of change of another variable (which may be more easily measured). The procedure of solving a related rates problem is to find an equation that relates two quantities and then use the chain rule to differentiate both sides with respect to time. Then, from knowing the rate of change of one value at a point in time, we can calculate the rate of change of another quantity at that moment in time. The equation which describes the application is derived from the verbal description and is called a mathematical model of the problem. For these equations we can relate different rates of change by using the chain rule and implicit differentiation.

Find a Related Rate

Given a function $y = f(x)$ where both x and y are functions of time t, we have $x = x(t)$ and $y = y(t)$; and so we can use the chain rule and implicit differentiation to determine $\frac{dy}{dt}$ and $\frac{dx}{dt}$. Knowing one of these values we can calculate the other.

Example (Solving Problems Involving Related Rates) Find the missing rates of change.

(a) Find $\frac{dy}{dt}$ when $x = 7$ given $y = 5\sqrt{x+9}$ and $\frac{dx}{dt} = 2$.

Solution. We have $\frac{dy}{dt} = \frac{5}{2\sqrt{x+9}}\frac{dx}{dt}$ and so when $x = 7$, $\frac{dy}{dt} = \frac{5}{2\sqrt{7+9}}(2) = \frac{5}{4}$.

(b) Find $\frac{dy}{dt}$ when $x = 1$ given $5xy = 10$ and $\frac{dx}{dt} = -2$.

Solution. We have $x\frac{dy}{dt} + y\frac{dx}{dt} = 0$ and so when $x = 1$, then $y = 2$ and $(1)\frac{dy}{dt} + (2)(-2) = 0$. Therefore, $\frac{dy}{dt} = 4$. ■

Problem Solving with Related Rates

Every related rates problem has a general situation (properties that hold true at every instant in time) and a specific situation (properties that hold true at a particular instant in time). Distinguishing between these two situations is often the key to successfully solving a related-rates problem.

Guidelines for Solving Related Rate Problems

(i) Read the problem carefully and identify all given quantities and unknown quantities to be determined. Introduce notation, make a sketch, and label the quantities.

(ii) Write equations involving the variables whose rates of change either are given or are to be determined.

(iii) Use implicit differentiatiation, by differentiating both sides with respect to time.

(iv) Substitute known variables and known rates of change into the resulting equation to determine the missing rate of change.

Example (Related Rates with Cones) Model a water tank by a cone 40 ft high with a circular base of radius 20 ft at the top. Water is flowing into the tank at a constant rate of $80\ \text{ft}^3/\text{min}$. How fast is the water level rising when the water is 12 feet deep?

Solution. Let x be the radius of the top circle of the body of water and y its height. The radius of the top circle is 20, the height of the cone is 40 ft. By similar triangles, $20/40 = x/y$ and so $x = \frac{1}{2}y$. The volume of the body of water is $V = \frac{1}{3}\pi x^2 y = \frac{1}{12}\pi y^3$. Then $\frac{dV}{dt} = \frac{1}{4}\pi y^2 \frac{dy}{dt}$. When $y = 12$, $x = 6$, and $\frac{dV}{dt} = 80$, so $80 = \frac{1}{4}\pi(12)^2\frac{dy}{dt}$ and thus $\frac{dy}{dt} = \frac{80(4)}{\pi(144)} \approx 0.71\ \text{ft}/\text{min}$. ■

Example (Related Rates with Spheres) Air is being pumped into a spherical balloon at a rate of 4.5 cubic feet per minute. Find the rate of change of the radius when the radius is 2 feet.

Solution. Let V be the volume of the balloon and let r be the radius. Because the volume is increasing at a rate of 4.5 cubic feet per minute, we know that at time t the rate of change of the volume is $\frac{dV}{dt} = \frac{9}{2}$. The equation that relates the radius r to the volume V is $V = \frac{4}{3}\pi r^3$. So we have $\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$ and by solving for $\frac{dr}{dt}$ we have $\frac{dr}{dt} = \frac{1}{4\pi r^2}\left(\frac{dV}{dt}\right)$. Finally, when $r = 2$, the rate of change of the radius is $\frac{dr}{dt} = \frac{1}{16\pi}\left(\frac{9}{2}\right) \approx 0.09$ foot per minute. ■

Example (Related Rates with Circles) A pebble is dropped into a calm pond, causing ripples in the form of concentric circles. The radius of the outer ripple is increasing at a constant rate of 1 foot per second. When the radius is 4 feet, at what rate is the total area A of the disturbed water changing?

Solution. For the area of a circle we use $A = \pi r^2$ and we differentiate implicitly with respect to time t leading to $\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$ by using the chain rule. Since $\frac{dr}{dt} = 1$ and when

$r = 4$ we have $\dfrac{dA}{dt} = 2\pi(4)(1) = 8\pi$. ■

Example (Related Rates with Right Triangles) Solve the related rate problem.

(a) A ladder 10 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 ft/s, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 ft from the wall?

Solution. Let x meters be the distance from the bottom of the ladder to the wall and y meters the distance from the top of the ladder to the ground. Since x and y are functions of time and $\dfrac{dx}{dt} = 1$ ft/s we are asked to find $\dfrac{dy}{dt}$ when $x = 6$ ft. The relationship between x and y is the Pythagorean Theorem, namely $x^2 + y^2 = 100$. Using implicit differentiation and the chain rule we have $2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt} = 0$. By solving for $\dfrac{dy}{dt}$ we have $\dfrac{dy}{dt} = -\dfrac{x}{y}\dfrac{dx}{dt}$. When $x = 6$ then $y = 8$ and so $\dfrac{dy}{dt} = \dfrac{-6}{8}(1) = \dfrac{-3}{4}$ ft/s.

(b) Car A is going west at 50 mi/h and car B is going north at 60 mi/h. Both are headed for the intersection of the two roads. At what rate are the cars approaching each other when car A is 0.3 mi and car B is 0.4 mi from the intersection?

Solution. At a given time t, let x be the distance from car A to the point of intersection P and let y be the distance from car B to P and let z be the distance between the cars, where x, y, and z are measured in miles. Since x and y are decreasing we take the derivatives to be negative and so we are given $\dfrac{dx}{dt} = -50$ mi/h and $\dfrac{dy}{dt} = -60$ mi/h. To find $\dfrac{dz}{dt}$ we use the Pythagorean Theorem, namely $x^2 + y^2 = z^2$ and differentiate with respect to time t. We have $2z\dfrac{dz}{dt} = 2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt}$ and by solving for $\dfrac{dz}{dt}$ we have $\dfrac{dz}{dt} = \dfrac{1}{z}\left(x\dfrac{dx}{dt} + y\dfrac{dy}{dt}\right)$. Now when $x = 0.3$ mi and $y = 0.4$ mi we have $z = 0.5$ mi and so

$$\frac{dz}{dt} = \frac{1}{0.5}[0.3(-50) + 0.4(-60)] = -78 \text{ mi}/h$$

Therefore, the cars are approaching each other at a rate of 78 mi/h. ■

Example (Related Rates with Similar Triangles) A person 6 ft tall walks away from a streetlight at the rate of 5 ft/s. If the light is 18 ft above ground level, how fast is the person's shadow lengthening?

Solution. Let x be the length of the shadow and y be the distance of the person from the street light. Using similar triangles, $\dfrac{x}{6} = \dfrac{x+y}{18}$ and by solving for y we have $y = 2x$. Thus, $dy/dt = 2(dx/dt)$ and given that $dy/dt = 5$ ft/s we find that $dx/dt = 2.5$ ft/s is the rate the shadow is lengthening. ■

Example (Related Rates with Triangles) At noon, a ship sails due north from a point P at 8 knots. Another ship, sailing at 12 knots, leaves the same point 1 h later on a course $60°$ east of north. How fast is the distance between the ships increasing at 5 P.M.?

Solution. Let A be the distance travelled by the first ship, and B for the distance travelled by the second ship, D for the distance between them, and θ the constant angle of $60°$. We need to find $\dfrac{dD}{dt}$ at $t = 5$. The equation that relates all the variables is the law of cosines and is

$$D^2 = A^2 + B^2 - 2AB\cos 60°.$$

Using the chain rule and implicit differentiation we have

$$2D\frac{dD}{dt} = 2A\frac{dA}{dt} + 2B\frac{dB}{dt} - 2\left(A\frac{dB}{dt} + B\frac{dA}{dt}\right)\left(\frac{1}{2}\right)$$

since $\cos 60° = \dfrac{1}{2}$. Then at $t = 5$, $A = 5(8) = 40$, $B = 12(4) = 48$, $\dfrac{dA}{dt} = 8$, $\dfrac{dB}{dt} = 12$, and

$$D = \sqrt{40^2 + 48^2 - 2(40)(48)\left(\frac{1}{2}\right)} = \sqrt{1984}.$$

So we have,

$$\frac{dD}{dt} = \frac{2(40)(8) + 2(48)(12) - (40)(12) - (48)(8)}{2\sqrt{1984}} = \frac{58}{\sqrt{31}} \approx 10.4171 \text{ knots}.$$

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