Separable Equations

A separable differential equation is one of the form $\frac{dy}{dx} = f(y)g(x)$

To solve it we separate it so on one side we just have x's & on the other side we have y's.

$$\int \frac{dy}{f(y)} = \int g(x)\,dx$$

Integrate both sides & add a "+ C" on the side with the independent variable (in this case x)

Solve for y if you can.

Although the process is straight forward, it is equally easy to mess these up.

Examples from Differential Equations by Edwards & Penney

Ex 1 Solve explicitly for y

$$\frac{dy}{dx} = (64xy)^{1/3}$$

* First we need to get this as a function of x multiplied by a function of y -- the 1/3 is messing us up

$$\frac{dy}{dx} = 64^{1/3}x^{1/3}y^{1/3}$$

$$\frac{dy}{dx} = 4x^{1/3}y^{1/3}$$

* Separate

$$\int \frac{dy}{y^{1/3}} = \int 4x^{1/3}\,dx$$

$$\int y^{-1/3}\,dy = \int 4x^{1/3}\,dx$$

$$\frac{3}{2}y^{2/3} = 4\left(\frac{3}{4}\right)x^{4/3} + C$$

$$\frac{3}{2} y^{2/3} = 3x^{4/3} + C$$

* Here is where people start to mess up. To solve for y, we need to get y b itself

$$y^{2/3} = \frac{2}{3}(3x^{4/3}) + \underbrace{\frac{2}{3}C}$$

C is just a constant, so $\frac{2}{3}C$ is another constant

$$y^{2/3} = 2x^{4/3} + C_1$$

$$\boxed{y = (2x^{4/3} + C_1)^{3/2}}$$

Note! $y = (2x^{4/3})^{3/2} + C_1^{3/2}$ is very very wrong

Likewise so is

$$y = (2x^{4/3})^{3/2} + C_1$$

Ex 2 Solve the IVP

$$2y\frac{dy}{dx} = \frac{x}{\sqrt{x^2-16}} \qquad y(5)=2$$

* Separate

$$\int 2y\,dy = \int \frac{x}{\sqrt{x^2-16}}\,dx$$

* Need to do a u-sub

$$u = x^2-16$$

$$du = 2x\,dx$$

$$\frac{1}{2}du = x\,dx$$

$$\int 2y\,dy = \int \frac{1}{2}\frac{1}{\sqrt{u}}\,du$$

$$\int 2y\,dy = \int \frac{1}{2}u^{-1/2}\,du$$

$$y^2 = u^{1/2} + C$$

$$y^2 = \sqrt{x^2-16} + C$$

$$y = \pm\sqrt{\sqrt{x^2-16} + C}$$

Do determine if we want the + or the −, we need to look at the initial condition $y(5)=2$. Because we have $+2$, we'll use the +

$$y=\sqrt{\sqrt{x^2-16}+C}$$

$$2=\sqrt{\sqrt{25-16}+C}$$

$$2=\sqrt{\sqrt{9}+C}$$

$$2=\sqrt{3+C} \longrightarrow C=1$$

$$\boxed{y=\sqrt{\sqrt{x^2-16}+1}}$$

* Again, note that when we take a square root of both sides, we can't choose what gets a square root, it needs to be the <u>whole</u> right side.

Ex 3 $\quad x\frac{dy}{dx} - y = 2x^2y \quad y(1) = 1$

* We have a little work to do before it is in the right form

$$x\frac{dy}{dx} = 2x^2y + y$$

$$\frac{dy}{dx} = \frac{2x^2y + y}{x}$$

$$= y\left(\frac{2x^2+1}{x}\right)$$

$$= y\left(2x + \frac{1}{x}\right)$$

$$\int \frac{dy}{y} = \int 2x + \frac{1}{x}\, dx$$

$$\ln|y| = x^2 + \ln|x| + C$$

$$|y| = e^{x^2 + \ln|x| + C}$$

$$|y| = e^{x^2 + \ln|x|}e^{C}$$

* or if we like being fancy

$$|y| = e^{x^2}e^{\ln|x|}e^{C}$$

$y = e^{x^2} \times K$

remember
$k = \pm e^{c}$, you don't need to write what k equals on your test

$y = kx e^{x^2}$

$y(1) = 1$

$1 = ke$

$k = \frac{1}{e}$

$\boxed{y = \frac{1}{e} x e^{x^2}}$ or $y = x e^{x^2 - 1}$

<u>Ex 4</u> $\frac{dy}{dx} = 6e^{2x-y}$ $y(0) = 0$

* we need to get this in the correct form

$\frac{dy}{dx} = 6e^{2x} e^{-y}$

$$\frac{dy}{dx} = 6e^{2x}e^{-y}$$

$$\int \frac{dy}{e^{-y}} = \int 6e^{2x}\,dx$$

$$\int e^{y}\,dy = \int 6e^{2x}\,dx$$

$$e^{y} = \frac{6}{2}e^{2x} + C$$

$$e^{y} = 3e^{2x} + C$$

$$y = \ln(3e^{2x} + C)$$

$y(0) = 0$

$$0 = \ln(3e^{0} + C)$$

* We know $\ln 1 = 0$, so we need to set $3e^{0} + C = 1$

$$3 + C = 1 \rightarrow C = -2$$

$$\boxed{y = \ln(3e^{2x} - 2)}$$