Sequences & Series

<u>Def</u>: A sequence is an ordered list of numbers $\{a_1, a_2, a_3, \ldots\}$

<u>Def</u>: A series, $\sum a_n$, is the sum of an infinite sequence.

Geometric Series

$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \ldots$

convergent if $|r| < 1$ <u>and</u> converges to $\frac{a}{1-r}$

divergent otherwise

Harmonic Series

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. We also know any constant multiple of the Harmonic series diverges

Divergence Test

$\sum a_n$ diverges if $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$

Telescoping Sum

* a_n is a rational function, express using partial fractions, look at the 1st few partial sums for a pattern.
* Alternatively, a_n is the difference of 2 functions, write out the 1st few partial sums & look for a pattern.

Ex 1 $a_n = \cos\left(\frac{1}{n^2+1}\right)$

a) Does the sequence $\{a_n\}$ converge or diverge?

b) Does $\sum_{n=1}^{\infty} a_n$ converge or diverge?

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a) $\lim_{n\to\infty} \cos\left(\frac{1}{n^2+1}\right) = \cos 0 = 1$

$\boxed{\text{Sequence converges to } 1}$

b) $\sum_{n=1}^{\infty} a_n$ diverges by the divergence test since $\lim_{n\to\infty} a_n = 1 \neq 0$

Ex 2 $a_n = \frac{-1}{7n}$

a) Does the sequence $\{a_n\}$ converge or diverge? b) Does the series converge or diverge?

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a) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{-1}{7n} = 0$

The sequence converges to 0

b) $\sum_{n=1}^{\infty} \frac{-1}{7n}$ the series diverges since it is a constant multiple of the Harmonic series

* NOTE: $\lim_{n\to\infty} a_n = 0$ DOES NOT imply $\sum a_n$ converges. If the Divergence Test doesn't work try another test.

Ex 3 $a_n = \frac{(-3)^n}{(20)^{n+1}}$ Same questions as before.

a) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{20} \cdot \left(\frac{-3}{20}\right)^n = 0$

The sequence converges to zero

b) $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(-3)^n}{20^{n+1}} = \frac{-3}{20^2} + \frac{(-3)^2}{20^3} + \frac{(-3)^3}{20^4} + \cdots$

$a + ar + ar^2 + \cdots$

$a = \frac{-3}{20^2}$ $r = \frac{-3}{20}$

$|r| = \left|\frac{-3}{20}\right| < 1$ so this is a convergent

Geometric series

$\sum_{n=1}^{\infty} a_n = \frac{a}{1-r} = \frac{-3/20^2}{1-\left(\frac{-3}{20}\right)} = \frac{-3/20^2}{\frac{23}{20}} = \frac{-3}{\cancel{20^2}} \cdot \frac{\cancel{20}}{23}$

$= \boxed{\frac{-3}{460}}$

Ex 4 $a_n = \frac{4}{n(n+1)}$ Same questions

a) $\lim_{n\to\infty} \frac{4}{n(n+1)} = 0$ Sequence converges to 0.

b) $\sum_{n=1}^{\infty} \frac{4}{n(n+1)} = \sum_{n=1}^{\infty} \frac{A}{n} + \frac{B}{n+1}$

$A(n+1) + Bn = 4$

$An + A + Bn = 4$

$A = 4$

$A + B = 0$

$B = -4$

$$\sum_{n=1}^{\infty} \frac{4}{n} - \frac{4}{n+1}$$

$$S_1 = \frac{4}{1} - \frac{4}{2}$$

$$S_2 = 4 - \cancel{\frac{4}{2}} + \cancel{\frac{4}{2}} - \frac{4}{3}$$

$$S_3 = 4 - \cancel{\frac{4}{3}} + \cancel{\frac{4}{3}} - \frac{4}{4}$$

$$S_4 = 4 - \cancel{\frac{4}{4}} + \cancel{\frac{4}{4}} - \frac{4}{5}$$

$\vdots$

$$S_n = 4 - \frac{4}{n+1}$$

$$\lim_{n\to\infty} S_n = 4$$

$\boxed{\text{Series converges to 4}}$

EX 5 Same question $a_n = \ln\left(\frac{n}{n+1}\right)$

a) $\lim_{n\to\infty} \ln\left(\frac{n}{n+1}\right) = \ln\left(\lim_{n\to\infty} \frac{n}{n+1}\right) = \ln(1) = 0$

Sequence converges to 0

b) $\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right) = \sum_{n=1}^{\infty} \ln(n) - \ln(n+1)$

$S_1 = \ln 1 - \ln 2 = -\ln 2$, $S_2 = \cancel{\ln 2} + \cancel{\ln 2} - \ln 3$,

$S_3 = -\ln 3 + \ln 3 - \ln 4 = -\ln 4$

$\vdots$

$S_n = -\ln(n+1)$

$\lim_{n\to\infty} S_n = -\infty$ Series diverges