Springs Revisited

The motion of an object with mass m on the end of a spring is given by

$$mx'' + bx' + kx = F_{ext}$$

Where m = mass of object
b = damping constant
k = spring constant
$x(t)$ = position of mass at time t
F_{ext} = external force

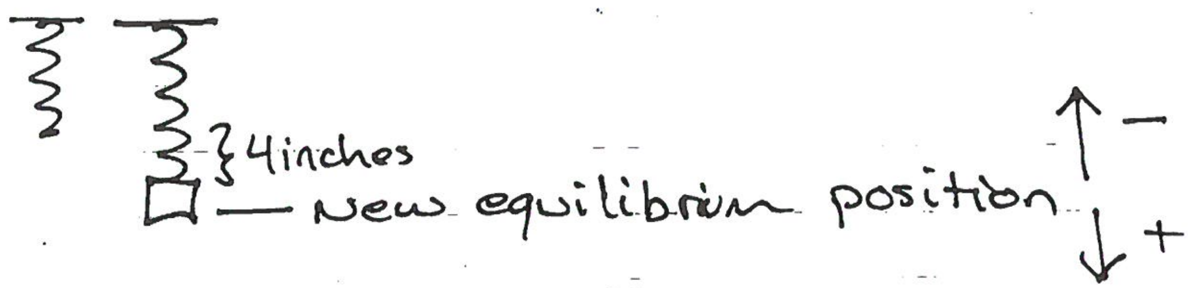
If $b = 0$: simple harmonic motion

If $b^2 - 4mk > 0$: overdamping
$b^2 - 4mk = 0$: critical damping
$b^2 - 4mk < 0$: underdamping

Example from Advanced Engineering Mathematics by Zill & Cullen

Ex 1 A 24-pound weight attached to the end of a spring, stretches it 4 inches. Find the equation of motion

If the weight is released from rest from a point 3 inches above the equilibrium position



$$mx'' + bx' + kx = F_{ext}$$

No damping, no external force
($b=0$) ($F_{ext}=0$)

$$mx'' + kx = 0$$

* We need to find m & k. Either they will give us a weight & we find mass or they will give us the mass outright in kg or slugs (mass units for English system)

* We know $W = 24\,lb$. From Newton's 2nd Law $F=ma \rightarrow W=mg$ ← acceleration due to gravity

Since we have English units (ft, lbs, etc)
$g = 32\,ft/s^2$ (I will give you gravity)

So $24 = m\,32$

$$m = \frac{24}{32} = \frac{3}{4}$$

* Now we just need k. Unless it is just given to you, we will find it using Hooke's Law $F = kx$

F is the force stretching the spring & x is how far it is stretched relative to the natural length.

* Weight of the object is our force.

$$W = k\left(\tfrac{1}{3}ft\right)$$

$4\text{ inches} = \frac{4}{12}ft = \frac{1}{3}ft$

* Remember our units need to agree lbs & ft or N & meters *

$$24 = k\,\tfrac{1}{3}$$

$$k = 72$$

$\frac{3}{4}x'' + 72x = 0$

* We need our initial conditions

$x(0) = -\frac{1}{4}$ ft

above equilibrium is negative

3 inches $= \frac{3}{12} = \frac{1}{4}$ ft

$x'(0) = 0$

released from rest means no initial velocity

If we were just asked to formulate the IVP

$$\boxed{\frac{3}{4}x'' + 72x = 0 \qquad x(0) = -\frac{1}{4} \quad x'(0) = 0}$$

For this first one we'll solve it all the way:

Characteristic equation $\frac{3}{4}r^2 + 72 = 0$

$$r^2 = \frac{-72(4)}{3}$$

$$r^2 = -24(4)$$

$$r = \pm\sqrt{-24 \cdot 4}$$

$$= \pm\sqrt{-4 \cdot 6 \cdot 4}$$

$$= \pm 4\sqrt{-6}$$

Case 3 $\longrightarrow$ $= \pm 4\sqrt{6}\, i$

$\alpha = 0 \quad \beta = 4\sqrt{6}$

$$x(t) = e^{0t}\left[c_1\cos(4\sqrt{6}t) + c_2\sin(4\sqrt{6}t)\right]$$
$$x(t) = c_1\cos(4\sqrt{6}t) + c_2\sin(4\sqrt{6}t)$$

$$x(0) = -1/4 = c_1\overset{1}{\cos 0} + \cancel{c_2\sin 0}$$
$$c_1 = -1/4$$

$$x(t) = -\tfrac{1}{4}\cos(4\sqrt{6}t) + c_2\sin(4\sqrt{6}t)$$

$$x'(t) = -\tfrac{1}{\cancel{4}}(-\sin(4\sqrt{6}t)\cancel{4}\sqrt{6}) + 4\sqrt{6}c_2\cos(4\sqrt{6}t)$$

$$x'(t) = \sqrt{6}\sin(4\sqrt{6}t) + 4\sqrt{6}c_2\cos(4\sqrt{6}t)$$

$$x'(0) = \cancel{\sqrt{6}\sin 0} + 4\sqrt{6}c_2\overset{1}{\cos 0} = 0$$
$$c_2 = 0$$

$$\boxed{x(t) = -\tfrac{1}{4}\cos(4\sqrt{6}t)}$$

simple harmonic motion, $b=0$

Examples from <u>Differential Equations</u> by Brannan & Boyce

Ex 2 A mass of .1 kg stretches a spring 5 cm. If the mass is set in motion from its equilibrium position with a downward velocity of 10 cm/s, and if there is no damping, determine the position x of the mass at time t.

Again $b=0$, $F_{ext}=0$

$$mx''+kx=0$$

We know $m=.1$ kg, we just need to find k using $F=kx$.
The force that stretches the spring is the weight of the object

$$W=kx$$
$$mg=kx$$
$$.1(9.8\,m/s^2)=k(.05\,m)$$

5cm = .05m

gravity if we are using the metric system

$$.98=k\tfrac{5}{100}$$
$$\frac{98}{5}=k$$

$$\boxed{.1x''+\tfrac{98}{5}x=0 \qquad x(0)=0 \qquad x'(0)=.1\,m/s}$$

equilibrium

IVP describing motion of mass

downward velocity + 10cm/s = .1m/s

$.1r^2 + \frac{98}{5} = 0$

$r^2 = -\frac{980}{5} = -196$

$r = \pm\sqrt{-196} = \pm 14i \quad \alpha = 0$

$\beta = 14$

$X(t) = e^{0t}[C_1\cos(14t) + C_2\sin(14t)]$

$X(t) = C_1\cos(14t) + C_2\sin(14t)$

$X(0) = 0 = C_1\cos 0 + \cancel{C_2\sin 0}$

$= C_1 \cdot 1$

$C_1 = 0$

$X(t) = C_2\sin(14t)$

$X'(t) = 14C_2\cos(14t)$

$X'(0) = 14C_2 = .1$

$C_2 = .1/14 = 1/140$

$$\boxed{X(t) = \frac{1}{140}\sin(14t)}$$

Ex 3 A mass weighing 16lb stretches a spring 3in. The mass is attached to a viseous damper with a damping constant of 2. If the mass is set in motion from its equilibrium position with a downward velocity of 3inches/sec find its position x(t) at time t. Determine when the mass first returns to its equilibrium position

$$mx'' + bx' + kx = F_{ext}$$

$W = 16\text{ lb} = mg \leftarrow$ they gave us weight we need mass

$$16 = m(32)$$

$$\boxed{m = \frac{16}{32} = 1/2}$$

$\boxed{b = 2} \leftarrow$ they gave this to us

* still need k $\quad F = kx$

$$W = 16 = k\left(\tfrac{1}{4}\right)$$

3inches $= \frac{1}{4}$ft

$$\boxed{k = 64}$$

IVP describing motion of mass

$$\boxed{\tfrac{1}{2}x'' + 2x' + 64x = 0 \qquad x(0)=0,\quad x'(0) = 1/4\ \text{ft/s}}$$

3 inches/s

* Remember downward = positive
upward = negative
likewise compression = negative
stretch = positive

$$\tfrac{1}{2}r^2 + 2r + 64 = 0$$

$$r^2 + 4r + 128 = 0$$

$$r = \frac{-4 \pm \sqrt{16 - 4(128)}}{2}$$

$$= \frac{-4 \pm \sqrt{16 - 4\cdot(4\cdot 32)}}{2}$$

$$= \frac{-4 \pm \sqrt{16 - 16\cdot 32}}{2}$$

$$= \frac{-4 \pm \sqrt{16(1-32)}}{2}$$

$$= \frac{-4 \pm 4\sqrt{-31}}{2} = -2 \pm 2\sqrt{31}\, i$$

$b^2 - 4mk < 0$
Underdamping
lots of oscillations

$$x(t) = e^{-2t}\left[c_1 \cos(2\sqrt{31}t) + c_2 \sin(2\sqrt{31}t)\right]$$

$$x(0) = 0 \rightarrow c_1 = 0$$

$$x(t) = e^{-2t} c_2 \sin(2\sqrt{31}t)$$

$$x'(t) = -2e^{-2t} c_2 \sin(2\sqrt{31}t) + e^{-2t} c_2 2\sqrt{31} \cos(2\sqrt{31}t)$$

$$x'(0) = \cancel{-2 \cdot 1 \cdot c_2 \cdot 0} + c_2 2\sqrt{31} = \frac{1}{4}$$

$$c_2 = \frac{1}{8\sqrt{31}}$$

$$\boxed{x(t) = \frac{1}{8\sqrt{31}} e^{-2t} \sin(2\sqrt{31}t)}$$

When will the mass return to equilibrium?

$$x(t) = 0 = \underline{\frac{1}{8\sqrt{31}} e^{-2t}} \sin(2\sqrt{31}t)$$

All we need is to know when $\sin(2\sqrt{31}t) = 0$ since the rest is never zero.

Obviously $t=0$ works, this makes sense since we are starting from equilibrium

We just want the next value of t that will work:

$\sin 0 = 0$, $\sin \pi = 0$, $\sin 2\pi$, etc

next value to give us zero

When does $2\sqrt{31}\, t = \pi$?

$$\boxed{t = \frac{\pi}{2\sqrt{31}}}$$

time to return to equilibrium.