Taylor & Maclaurin Series

Taylor series: Centered at a

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)(x-a)^n}{n!}$$

$$= f(a) + \frac{f'(a)(x-a)}{1!} + \frac{f''(a)(x-a)^2}{2!} + \cdots$$

Maclaurin series:

(Taylor series w/ $a=0$)

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)\, x^n}{n!}$$

$$= f(a) + \frac{f'(a)x}{1!} + \frac{f''(a)x^2}{2!} + \frac{f'''(a)x^3}{3!} + \cdots$$

Technique:

1. Find f', f'', f''', f^4
2. Evaluate $f(a)$, $f'(a)$, $f''(a)$, $f'''(a)$, $f^4(a)$
3. Plug into

$$f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \frac{f'''(a)(x-a)^3}{3!} + \cdots$$

4. Try to find a condensed form of 3, $\sum_{n=0}^{\infty}$ ___

Sometimes you can skip step 3

Ex 1

a) Derive a Maclaurin series for cos x

Step 1: Take derivatives

$f(x) = \cos x$

$f'(x) = -\sin x$

$f''(x) = -\cos x$

$f'''(x) = \sin x$

$f^4(x) = \cos x$

* I like to take 4 derivatives because often that is sufficient to see a pattern, but you can take more.

Step 2: Plug in a

* Maclaurin series $\to a = 0$

$f(0) = \cos 0 = 1$

$f'(0) = -\sin 0 = 0$

$f''(0) = -\cos 0 = -1$

$f'''(0) = \sin 0 = 0$

$f^4(0) = \cos 0 = 1$

Step 3: Plug into $f(a) + \frac{f'(a)(x-a)}{1!} + \frac{f''(a)(x-a)^2}{2!} + \cdots$

$$\cos x = 1 + 0x - \frac{1x^2}{2!} + \frac{0x^3}{3!} + \frac{1x^4}{4!} + \cdots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$

Step 4: Condense

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$ ← even powers

b) Find the Maclaurin of $\cos 3x$.

* We already have the Maclaurin for

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

In general, $\cos(\ ) = \sum_{n=0}^{\infty} \frac{(-1)^n (\ )^{2n}}{(2n)!}$

so $\cos(3x) = \sum_{n=0}^{\infty} \frac{(-1)^n (3x)^{2n}}{(2n)!}$

$$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n} x^{2n}}{(2n)!}}$$

c) Use series to evaluate

$$\int \frac{1}{x^2} \cos(3x)\, dx$$

$$= \int \frac{1}{x^2} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n} 3^{2n}}{(2n)!}\, dx$$

* Bring to the inside of the sum

$$= \int x^{-2} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n} 3^{2n}}{(2n)!}\, dx$$

$$= \int \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n-2} 3^{2n}}{(2n)!}\, dx$$

* When we integrate, everything that is not an x gets treated as a constant. Remember $\int x^n dx = \frac{x^{n+1}}{n+1} + C$ $n \neq -1$

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n-2+1} 3^{2n}}{(2n)!\,(2n-2+1)} + C = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n-1} 3^{2n}}{(2n)!\,(2n-1)} + C}$$

$\boxed{\text{Ex 2}}$ Find a Taylor series for $f(x) = \sin x$ centered at $a = \pi/2$

→ Derivatives

$f(x) = \sin x$

$f'(x) = \cos x$

$f''(x) = -\sin x$

$f'''(x) = -\cos x$

$f^4(x) = \sin x$

→ Plug in $a = \pi/2$ (Evaluate f & its derivatives at a)

$f(\pi/2) = \sin \pi/2 = 1$

$f'(\pi/2) = \cos \pi/2 = 0$

$f''(\pi/2) = -\sin \pi/2 = -1$

$f'''(\pi/2) = -\cos \pi/2 = 0$

$f^4(\pi/2) = \sin \pi/2 = 1$

→ Plug into $f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \frac{f'''(a)(x-a)^3}{3!} + \ldots$

$$1 + 0(x-\pi/2) - \frac{1(x-\pi/2)^2}{2!} + \frac{0(x-\frac{\pi}{2})^3}{3!} + \frac{1(x-\pi/2)^4}{4!} + \ldots$$

$$= 1 - \frac{(x-\pi/2)^2}{2!} + \frac{(x-\pi/2)^4}{4!} - \ldots$$

→ Condense, Notice! even powers so

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n (x-\pi/2)^{2n}}{(2n)!}$$

* This looks very different from our Maclaurin for $\sin x$ because $a = \pi/2$

Ex 3

a) Derive the Maclaurin for e^x

* This is by far the easiest Maclaurin series to find

→ Derivatives

$f(x)=e^x$

$f'(x)=e^x$

$\vdots$

$f^n(x)=e^x$

→ Plug in a $(a=0 \rightarrow \text{Maclaurin})$

$f(0)=e^0=1$

$f'(0)=e^0=1$

$f^n(0)=e^0=1$

→ $f(a)+f'(a)(x-a)+\frac{f''(a)(x-a)^2}{2!}+\frac{f'''(a)(x-a)^3}{3!}+\ldots$

$1+1x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots$

$=\sum_{n=0}^{\infty}\frac{x^n}{n!}$

b) Find the Maclaurin for $f(x)=e^{\sqrt{x}}$

* We know the Maclaurin for $e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}$

In general $e^{(\ )}=\sum_{n=0}^{\infty}\frac{(\ )^n}{n!}$ so

$e^{\sqrt{x}}=\sum_{n=0}^{\infty}\frac{(\sqrt{x})^n}{n!}=\sum_{n=0}^{\infty}\frac{(x^{1/2})^n}{n!}=\sum_{n=0}^{\infty}\frac{x^{n/2}}{n!}$

c) Use series to evaluate $\int e^{\sqrt{x}}dx$

$\int e^{\sqrt{x}}dx=\int\sum_{n=0}^{\infty}\frac{x^{n/2}}{n!}dx=\boxed{\sum_{n=0}^{\infty}\frac{x^{n/2+1}}{n!\,(\frac{n}{2}+1)}+C}$

$\boxed{\text{Ex 4}}$ Find the Taylor series for $f(x) = \frac{1}{x}$ where $a=1$

$\rightarrow f(x) = \frac{1}{x} = x^{-1}$ (easier to take derivatives in this form)

$f'(x) = -x^{-2}$

$f''(x) = 2x^{-3}$

$f'''(x) = -2 \cdot 3x^{-4}$

$f^4(x) = 2 \cdot 3 \cdot 4x^{-5}$

$\rightarrow f(1) = 1$

$f'(1) = -1$

$f''(1) = 2$

$f'''(1) = -2 \cdot 3$

$f^4(1) = 2 \cdot 3 \cdot 4$

$\rightarrow f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \ldots$

$$1 - 1(x-1) + \frac{\cancel{2}(x-1)^2}{\cancel{2!}} - \frac{\cancel{2 \cdot 3}(x-1)^3}{\cancel{3!}} + \frac{\cancel{2 \cdot 3 \cdot 4}(x-1)^4}{\cancel{4!}} + \ldots$$

$$= 1 - (x-1) + (x-1)^2 - (x-1)^3 + (x-1)^4 - \ldots$$

$\rightarrow \boxed{\frac{1}{x} = \sum_{n=0}^{\infty} (x-1)^n(-1)^n}$

the radius of convergence is 1 for this but they haven't asked us to find it

$\boxed{\text{Ex 5}}$ Find the Taylor series, $a=1$
$f(x)=\sqrt{x}$

$\rightarrow f(x)=x^{1/2}$

$f'(x)=\frac{1}{2}x^{-1/2}$

$f''(x)=-\frac{1}{4}x^{-3/2}$

$f'''(x)=\frac{3}{8}x^{-5/2}$

$f^4(x)=-\frac{3\cdot 5}{16}x^{-7/2}$

$f^5(x)=\frac{3\cdot 5\cdot 7}{32}x^{-9/2}$ ← extra derivative because pattern is not obvious

$\rightarrow f(1)=1$

$f'(1)=\frac{1}{2}$

$f''(1)=-\frac{1}{4}$

$f'''(1)=\frac{3}{8}$

$f^4(1)=-\frac{3\cdot 5}{16}$

$f^5(1)=\frac{3\cdot 5\cdot 7}{32}$

$\rightarrow f(a)+f'(a)(x-a)+\frac{f''(a)(x-a)^2}{2!}+\dots$

$2^3 \qquad 2^4 \qquad 2^5$

$1+\frac{1}{2}(x-1)-\frac{\frac{1}{4}(x-1)^2}{2!}+\frac{\frac{3}{8}(x-1)^3}{3!}-\frac{\frac{3\cdot 5}{\cancel{16}}(x-1)^4}{4!}+\frac{\frac{3\cdot 5\cdot 7}{\cancel{32}}(x-1)^5}{5!}+\dots$

* This resembles the Binomial series

→ Describe the easy stuff first

$$\sum_{n=?}^{\infty}\frac{(x-1)^n(-1)^{n+1}}{n!\,2^n}(\ ?\ )$$

← notice the first term doesn't alternate sign, this tells us our series can't start at $n=0$; however $(-1)^{n+1}$ seems to work for all other terms

We can use our k−n+1 trick from Binomial series to help figure out the pattern in the numerater

$$\frac{1}{2}-n+1 = \frac{3}{2}-n = \frac{1}{2}(3-2n) = -\frac{1}{2}(2n-3)$$

accounted for w/ $(-1)^{n+1}$

accounted for in the $\frac{1}{2^n}$ part

If $n=0$: $2\cdot 0-3<0$

$n=1$: $2\cdot 1-3<0$

$n=2$: $2\cdot 2-3>0$

start series where it changes sign

$2\cdot 2-3=1$

$2\cdot 3-3=3$

$2\cdot 4-3=5$

$$\boxed{\underset{n=0}{1}+\underbrace{\frac{1}{2}(x-1)}_{n=1}+\sum_{n=2}^{\infty}\frac{(-1)^{n+1}(x-1)^n\left(1\cdot 3\cdot 5\cdots(2n-3)\right)}{2^n n!}}$$