Trig Substitution

Useful in problems that have
$\sqrt{u^2+a^2}$, $\sqrt{u^2-a^2}$, or $\sqrt{a^2-u^2}$

The basic idea here is to substitute u for a trig function so that what's under the square root becomes a trig identity

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Some examples from Calculus by Edwards & Penney

If the integral involves	then substitute	and use the identity
a^2-u^2	$u = a\cdot\sin\theta$	$1-\sin^2\theta = \cos^2\theta$ $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$
a^2+u^2	$u = a\cdot\tan\theta$	$1+\tan^2\theta = \sec^2\theta$ $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$
u^2-a^2	$u = a\cdot\sec\theta$	$\sec^2\theta - 1 = \tan^2\theta$ $0 \le \theta < \frac{\pi}{2}$

Ex 1 $\int \frac{\sqrt{x^2-1}}{x^2}\,dx$

☆ This matches the form u^2-a^2 where $u=x$ & $a=1$ so $x=1\sec\theta$

$dx = \sec\theta\tan\theta\,d\theta$

- Plug into the integrand – don't forget to plug in for dx!

$$\int \frac{\sqrt{\sec^2\theta-1}}{\sec^2\theta}\sec\theta\tan\theta\,d\theta$$

- Simplify under the square root

$\sec^2\theta - 1 = \tan^2\theta$

$\sin^2\theta+\cos^2\theta=1$ so

$$\frac{\sin^2\theta}{\cos^2\theta}+\frac{\cos^2\theta}{\cos^2\theta}=\frac{1}{\cos^2\theta}$$

$$\tan^2\theta+1=\sec^2\theta$$

$$\tan^2\theta=\sec^2\theta-1$$

$$\int \frac{\sqrt{\tan^2\theta}}{\sec^2\theta}\sec\theta\tan\theta\,d\theta$$

$$= \int \frac{\tan\theta\sec\theta\tan\theta\,d\theta}{\sec^2\theta}$$

$$= \int \frac{\tan^2\theta}{\sec\theta}\,d\theta$$

$$= \int \frac{\sec^2\theta-1}{\sec\theta}\,d\theta$$

$$= \int \sec\theta - \frac{1}{\sec\theta}\,d\theta$$

$$= \int \sec\theta - \cos\theta\,d\theta$$

At this point it becomes a trig integral. We could switch over to sin & cos to get $\int \frac{\frac{\sin^2\theta}{\cos^2\theta}}{\frac{1}{\cos\theta}}\,d\theta = \int \frac{\sin^2\theta}{\cos^2\theta}\cos\theta\,d\theta$

$$= \int \frac{\sin^2\theta}{\cos\theta}\,d\theta = \int \frac{1-\cos^2\theta}{\cos\theta}\,d\theta$$

$$= \int \frac{1}{\cos\theta} - \frac{\cos^2\theta}{\cos\theta}\,d\theta$$

$$= \int \sec\theta - \cos\theta\,d\theta$$

but I'm going to do it more efficiently

$$\int \sec\theta - \cos\theta \, d\theta$$

$$= \ln|\sec\theta + \tan\theta| - \sin\theta + C$$

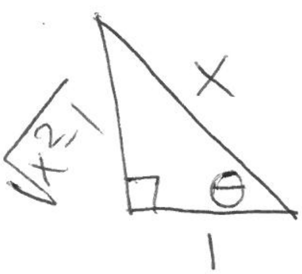
we've done $\int \sec\theta \, d\theta$ previously

- We need to get our final answer back in terms of x

we started out with $\frac{x}{1} = \sec\theta = \frac{hyp}{adj}$

$\frac{1}{\cos\theta}$

- Form a right triangle



& solve for the remaining side using the Pythagorean thm When you do this what'll be under the square root will match your original integral

Recall we had $\int \frac{\sqrt{x^2-1}}{x^2} \, dx$

$$= \ln|\sec\theta + \tan\theta| - \sin\theta + C$$

$$= \ln\left|\frac{hyp}{adj} + \frac{opp}{adj}\right| - \frac{opp}{hyp} + C$$

$$= \ln\left|\frac{x}{1} + \frac{\sqrt{x^2-1}}{1}\right| - \frac{\sqrt{x^2-1}}{x} + C$$

$$= \boxed{\ln\left|x + \sqrt{x^2-1}\right| - \frac{\sqrt{x^2-1}}{x} + C}$$

$\boxed{\text{Ex 2}}$ $\displaystyle\int \frac{dx}{(4x^2+9)^3}$

☆ This matches the form u^2+a^2 where $u=2x$ & $a=3$ so $u = a\tan\theta$ becomes $2x = 3\tan\theta$ or $x=\frac{3}{2}\tan\theta$
(You could also do this as a u-sub followed by a trig sub)

$x = \frac{3}{2}\tan\theta$
$dx = \frac{3}{2}\sec^2\theta\, d\theta$

° Plug in $\displaystyle\int \frac{\frac{3}{2}\sec^2\theta\, d\theta}{\left(4\left(\frac{3}{2}\tan\theta\right)^2+9\right)^3} = \int \frac{\frac{3}{2}\sec^2\theta\, d\theta}{\left(4\left(\frac{9}{4}\tan^2\theta\right)+9\right)^3}$

don't forget to square the number in front of your trig function

$$\int \frac{\frac{3}{2}\sec^2\theta\, d\theta}{(9\tan^2\theta+9)^3}$$

$$= \int \frac{\frac{3}{2}\sec^2\theta\, d\theta}{(9(\tan^2\theta+1))^3} = \int \frac{\frac{3}{2}\sec^2\theta\, d\theta}{(9\sec^2\theta)^3}$$

$$\int \frac{\frac{3}{2}\sec^2\theta \, d\theta}{9^3 \sec^6\theta} = \int \frac{\frac{3}{2}}{9^3} \frac{1}{\sec^4\theta} d\theta$$

$$\int \frac{3/2}{9^3} \cos^4\theta \, d\theta$$

cos is raised to an even power by itself we'll need the $\frac{1}{2}$ angle identities

$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$

$$\int \frac{3/2}{9^3} (\cos^2\theta)^2 d\theta = \int \frac{3/2}{9^3}\left(\frac{1}{2}(1+\cos 2\theta)\right)^2 d\theta$$

$$= \int \frac{3/2}{9^3}\left(\frac{1}{2}\right)^2 (1+\cos 2\theta)^2 d\theta$$

$$= \int \frac{1}{243} \cdot \frac{1}{8}\left[1 + 2\cos 2\theta + \cos^2 2\theta\right] d\theta$$

$u = 2\theta$

$du = 2d\theta$

$\frac{1}{2}du = d\theta$

I prefer u-sub before attempting 1/2 angle -- it is very easy to be off by a constant

$$\int \frac{1}{243} \cdot \frac{1}{8} \cdot \frac{1}{2}\left[1 + 2\cos u + \cos^2 u\right] du$$

$$\int \frac{1}{243} \cdot \frac{1}{16}\left[1 + 2\cos u + \frac{1}{2}(1+\cos 2u)\right] du$$

$$\frac{1}{243}\,\frac{1}{16}\left[u + 2\sin u + \frac{1}{2}\left(u + \frac{1}{2}\sin 2u\right)\right] + C$$

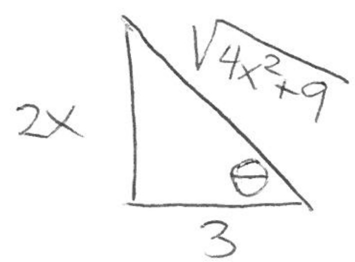
- Plug $u = 2\theta$ back in

$$\frac{1}{243}\,\frac{1}{16}\left[2\theta + 2\sin 2\theta + \frac{1}{2}\left(2\theta + \frac{1}{2}\sin 4\theta\right)\right] + C$$

- We need to get the final answer in terms of θ

$$x = \frac{3}{2}\tan\theta$$

$$\frac{2x}{3} = \tan\theta = \frac{opp}{adj}$$



$$\theta = \tan^{-1}\left(\frac{2x}{3}\right)$$

only do this if part of your answer is θ

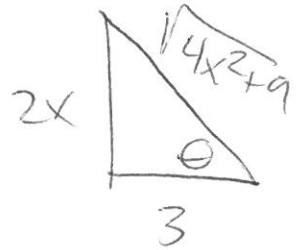
$$\frac{1}{243}\,\frac{1}{16}\left[2\tan^{-1}\left(\frac{2x}{3}\right) + 2\left(2\sin\theta\cos\theta\right) + \tan^{-1}\left(\frac{2x}{3}\right) + \frac{1}{4}\left(2\sin 2\theta\cos 2\theta\right)\right] + C$$

same idea

$$\underbrace{\sin 2\theta = 2\sin\theta\cos\theta}_{\text{I would give you this on a test}}$$

$$\frac{1}{243}\,\frac{1}{16}\left[3\tan^{-1}\left(\frac{2x}{3}\right)+4\sin\theta\cos\theta+\cancel{\frac{2}{4}}\underbrace{(2\cos\theta\sin\theta)}_{\sin 2\theta}\underbrace{(\cos^2\theta-\sin^2\theta)}_{\cos 2\theta}\right.+C$$

$$\frac{1}{243}\,\frac{1}{16}\left[3\tan^{-1}\left(\frac{2x}{3}\right)+4\sin\theta\cos\theta+(\cos\theta\sin\theta)(\cos^2\theta-\sin^2\theta)\right]+C$$



$$\sin\theta=\frac{opp}{hyp}$$

$$\frac{1}{243}\,\frac{1}{16}\left[3\tan^{-1}\left(\frac{2x}{3}\right)+4\left(\frac{2x}{\sqrt{4x^2+9}}\right)\left(\frac{3}{\sqrt{4x^2+9}}\right)+\left(\frac{3}{\sqrt{4x^2+9}}\,\frac{2x}{\sqrt{4x^2+9}}\right)\left(\frac{9}{4x^2+9}-\frac{4x^2}{4x^2+9}\right)\right]+C$$

$$\frac{1}{243}\,\frac{1}{16}\left[3\tan^{-1}\left(\frac{2x}{3}\right)+\frac{24x}{4x^2+9}+\frac{54x}{(4x^2+9)^2}-\frac{24x^3}{(4x^2+9)^2}\right]+C$$

$$\frac{1}{1296}\left[\tan^{-1}\left(\frac{2x}{3}\right)+\frac{8x}{4x^2+9}+\frac{18x}{(4x^2+9)^2}-\frac{8x^3}{(4x^2+9)^2}\right]+C$$

$$\frac{1}{1296}\left[\tan^{-1}\left(\frac{2x}{3}\right)+\frac{32x^3+72x+18x-8x^3}{(4x^2+9)^2}\right]+C$$

$$\frac{1}{1296}\left[\tan^{-1}\left(\frac{2x}{3}\right)+\frac{24x^3+90x}{(4x^2+9)^2}\right]+C$$

Not as much fun as I anticipated

$\boxed{\text{Ex 3}}$ $\displaystyle\int_0^{\sqrt{3}} \frac{1}{(4-t^2)^{5/2}}\,dt$

☆ This is the form a^2-u^2 where a $=2$ & $u=t$ so

$t = a\sin\theta$, $t = 2\sin\theta$

$dt = 2\cos\theta\, d\theta$

• We also need to adjust our bounds here we know $0 \le t \le \sqrt{3}$

we want the bounds for θ, $t = 2\sin\theta$

$0 = 2\sin\theta$

$0 = \frac{0}{2} = \sin\theta \rightarrow \theta = 0$

$\sqrt{3} = 2\sin\theta$

$\frac{\sqrt{3}}{2} = \sin\theta \rightarrow \theta = \pi/3$

$$\int_0^{\pi/3} \frac{2\cos\theta\, d\theta}{(4-4\sin^2\theta)^{5/2}} = \int_0^{\pi/3} \frac{2\cos\theta\, d\theta}{(4(1-\sin^2\theta))^{5/2}}$$

$$= \int_0^{\pi/3} \frac{2\cos\theta\, d\theta}{(4\cos^2\theta)^{5/2}} = \int_0^{\pi/3} \frac{2\cos\theta\, d\theta}{\left(\sqrt{4\cos^2\theta}\right)^5}$$

$$= \int_0^{\pi/3} \frac{2\cos\theta\, d\theta}{(2\cos\theta)^5} = \int_0^{\pi/3} \frac{1}{16} \frac{1}{\cos^4\theta}\, d\theta$$

$$\int_0^{\pi/3} \frac{1}{16} \frac{1}{\cos^4\theta}\, d\theta = \int_0^{\pi/3} \frac{1}{16} \sec^4\theta \, d\theta$$

$$= \int_0^{\pi/3} \frac{1}{16} \sec^2\theta \sec^2\theta \, d\theta$$

$$= \int_0^{\pi/3} \frac{1}{16} \left(\tan^2\theta + 1\right) \sec^2\theta \, d\theta$$

$u = \tan\theta$ $\quad u(0) = \tan 0 = 0$

$du = \sec^2\theta \, d\theta$ $\quad u(\pi/3) = \tan \pi/3 = \dfrac{\sin \pi/3}{\cos \pi/3} = \dfrac{\sqrt{3}/2}{\frac{1}{2}}$

$= \sqrt{3}$

$$\int_0^{\sqrt{3}} \frac{1}{16} (u^2 + 1)\, du$$

$$\frac{1}{16} \left[\frac{1}{3}u^3 + u\right]_0^{\sqrt{3}}$$

$$= \boxed{\frac{1}{16} \left[\frac{1}{3}(\sqrt{3})^3 + \sqrt{3}\right]}$$

$$= \frac{1}{16} \left[2\sqrt{3}\right]$$

$$= \frac{\sqrt{3}}{8}$$

these are our original bounds for t. If we had different original bounds like $\int_0^1 \frac{dt}{(4-t^2)^{5/2}}$ this wouldn't have happened

$$\int_0^{\pi/6} \frac{1}{16} (\tan^2\theta + 1) \sec^2\theta \, d\theta$$

$$= \int_0^{1/\sqrt{3}} \frac{1}{16} (u^2 + 1)\, du$$

Ex 4

Complete the square & evaluate

$$\int \frac{x}{\sqrt{x^2+4x+8}}\,dx$$

$$x^2+\underbrace{4x}+8 = (x+b)^2 + C$$
$$= x^2+\underbrace{2bx}+b^2+C$$

$4x = 2bx \quad b=2$

$$8 = b^2 + C$$
$$8 = 4 + C$$
$$C = 4$$

$(x+2)^2+4$

$$\int \frac{x}{\sqrt{(x+2)^2+4}}\,dx$$

* Form u^2+a^2

so $u = x+2 \quad a=2$, $u = a\tan\theta$

$x+2 = 2\tan\theta$

$dx = 2\sec^2\theta\, d\theta$

(You could also do a u-sub with $u = x+2$ & then get $\int \frac{u-2}{\sqrt{u^2+4}}\,du$ & do $u = 2\tan\theta$)

$$\int \frac{x \, 2\sec^2\theta \, d\theta}{\sqrt{(2\tan\theta)^2+4}}$$

need to plug in for θ

$x+2 = 2\tan\theta$

$x = 2\tan\theta - 2$

$$= \int \frac{(2\tan\theta - 2)\, 2\sec^2\theta \, d\theta}{\sqrt{4\tan^2\theta + 4}}$$

$$= \int \frac{(2\tan\theta - 2)\, \cancel{2}\sec^{\cancel{2}}\theta}{\cancel{\sqrt{4\sec^2\theta}}} \, d\theta$$

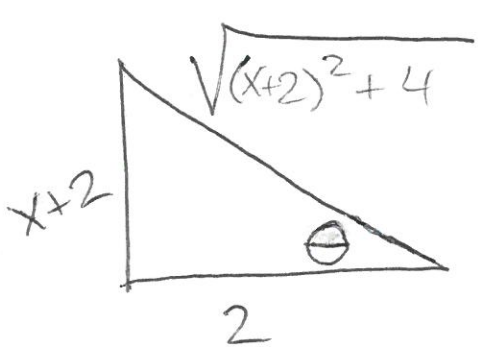
$$= \int 2\tan\theta\sec\theta - 2\sec\theta \, d\theta$$

$$= 2\sec\theta - 2\ln|\sec\theta + \tan\theta| + C$$

- Plug back in for x

$x+2 = 2\tan\theta$

$$\frac{x+2}{2} = \tan\theta = \frac{opp}{adj}$$



$= 2\sec\theta - 2\ln|\sec\theta + \tan\theta| + C$

$= 2\frac{hyp}{adj} - 2\ln\left|\frac{hyp}{adj} + \frac{opp}{adj}\right| + C$

$$= \boxed{\cancel{2}\frac{\sqrt{(x+2)^2+4}}{\cancel{2}} - 2\ln\left|\frac{\sqrt{(x+2)^2+4}}{2} + \frac{x+2}{2}\right| + C}$$

Ex 5 $\int e^x\sqrt{1-e^{2x}}\,dx = \int \underline{e^x}\sqrt{1-(e^x)^2}\,\underline{dx}$

$u = e^x \quad du = e^x dx$

$\int \sqrt{1-u^2}\,du$ ★ Form: $a^2 - w^2$

$u = w \quad a = 1$

using w since we already have a u in the problem

$u = 1\sin\theta$

$du = 1\cos\theta\,d\theta$

$\int \sqrt{1-\sin^2\theta}\,\cos\theta\,d\theta$

$= \int \sqrt{\cos^2\theta}\,\cos\theta\,d\theta$

$= \int \cos^2\theta\,d\theta$

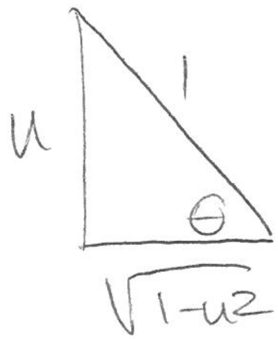
$\frac{1}{2}$ angle identity $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$

$$\int \frac{1}{2}(1+\cos 2\theta)\, d\theta$$

$$= \frac{1}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C$$

$u = \sin\theta$

$\frac{u}{1} = \frac{opp}{hyp}$



$$= \frac{1}{2}\left(\sin^{-1}(u) + \frac{1}{2}(2\sin\theta\cos\theta)\right) + C$$

$\sin 2\theta = 2\sin\theta\cos\theta$, I'd give you

$$= \frac{1}{2}\left(\sin^{-1}u + \frac{u}{1}\,\frac{\sqrt{1-u^2}}{1}\right) + C$$

$$= \boxed{\frac{1}{2}\left(\sin^{-1}(e^x) + e^x\sqrt{1-e^{2x}}\right) + C}$$

$\boxed{\text{Ex 6}}$ Find the arc length of $y = \ln x$ over the interval $[1,5]$

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

$$= \int_1^5 \sqrt{1 + \left[\frac{1}{x}\right]^2}\, dx$$

$$= \int_1^5 \sqrt{1 + \frac{1}{x^2}}\, dx$$

$$= \int_1^5 \sqrt{\frac{x^2}{x^2} + \frac{1}{x^2}}\, dx$$

$$= \int_1^5 \sqrt{\frac{x^2+1}{x^2}}\, dx$$

$$= \int_1^5 \frac{\sqrt{x^2+1}}{x}\, dx$$

$x = 1 \tan\theta$

$dx = \sec^2\theta\, d\theta$

$1 = \tan\theta \rightarrow \theta = \frac{\pi}{4}$

$5 = \tan\theta \rightarrow \theta = \tan^{-1}(5)$

$$\int_{\pi/4}^{\tan^{-1}5} \frac{\sqrt{\tan^2\theta + 1}}{\tan\theta} \sec^2\theta\, d\theta$$

$$\int_{\pi/4}^{\tan^{-1}5} \frac{\sqrt{\sec^2\theta}\ \sec^2\theta}{\tan\theta}\,d\theta$$

$$\int_{\pi/4}^{\tan^{-1}5} \frac{\sec^3\theta}{\tan\theta}\,d\theta$$

$$= \int_{\pi/4}^{\tan^{-1}5} \frac{\sec^3\theta}{\frac{\sin\theta}{\cos\theta}}\,d\theta = \int_{\pi/4}^{\tan^{-1}5} \frac{\sec^3\theta\cos\theta}{\sin\theta}\,d\theta$$

$$= \int_{\pi/4}^{\tan^{-1}5} \frac{\sec^2\theta}{\sin\theta}\,d\theta$$

$$= \int_{\pi/4}^{\tan^{-1}5} \frac{(1+\tan^2\theta)}{\sin\theta}\,d\theta$$

$$= \int_{\pi/4}^{\tan^{-1}5} \frac{1}{\sin\theta} + \frac{\frac{\sin^2\theta}{\cos^2\theta}}{\sin\theta}\,d\theta$$

$$= \int_{\pi/4}^{\tan^{-1}5} \csc\theta + \frac{\sin\theta}{\cos^2\theta}\,d\theta$$

I'd give you this

$$\int \csc\theta \cdot \left(\frac{\csc\theta + \cot\theta}{\csc\theta + \cot\theta}\right) d\theta$$

$$= \int \frac{\csc^2\theta + \csc\theta\cot\theta}{\csc\theta + \cot\theta} = \int -\frac{1}{u}\,du =$$

$$-\ln|\csc\theta + \cot\theta| + C$$

$$u = \csc\theta + \cot\theta$$

$$du = (-\csc\theta\cot\theta - \csc^2\theta)\,d\theta$$

$$\int \frac{\sin\theta}{\cos^2\theta}\,d\theta = \int -\frac{1}{u^2}\,du$$

$$= \frac{1}{u} + C$$

$u = \cos\theta$

$du = -\sin\theta\,d\theta$

$$= \frac{1}{\cos\theta} + C$$

$$= \sec\theta + C$$

you should be able to do this

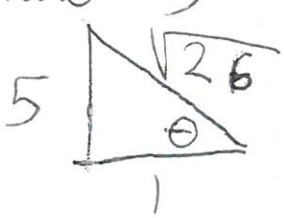
$$\int_{\pi/4}^{\tan^{-1}5} \csc\theta + \frac{\sin\theta}{\cos^2\theta}\,d\theta$$

$$= -\ln|\underbrace{\csc\theta}_{\frac{hyp}{opp}} + \underbrace{\cot\theta}_{\frac{adj}{opp}}| + \sec\theta \Big|_{\pi/4}^{\tan^{-1}5}$$

$$= -\ln|\csc(\tan^{-1}5) + \cot(\tan^{-1}5)| + \sec(\tan^{-1}5) - \left[-\ln\left(\csc\frac{\pi}{4} + \cot\frac{\pi}{4}\right) + \sec\frac{\pi}{4}\right]$$

$\theta = \tan^{-1}5$

$\tan\theta = 5$



$$= \boxed{-\ln\left|\frac{\sqrt{26}}{5} + \frac{1}{5}\right| + \sqrt{26} - \left(-\ln(\sqrt{2}+1) + \sqrt{2}\right)}$$