Volumes of Solids of Revolution (Washer examples)

Example: Find the volume of the region bounded by $y = x^2$, $y = x$ when it is revolved about:

a. the x-axis

b. the y-axis

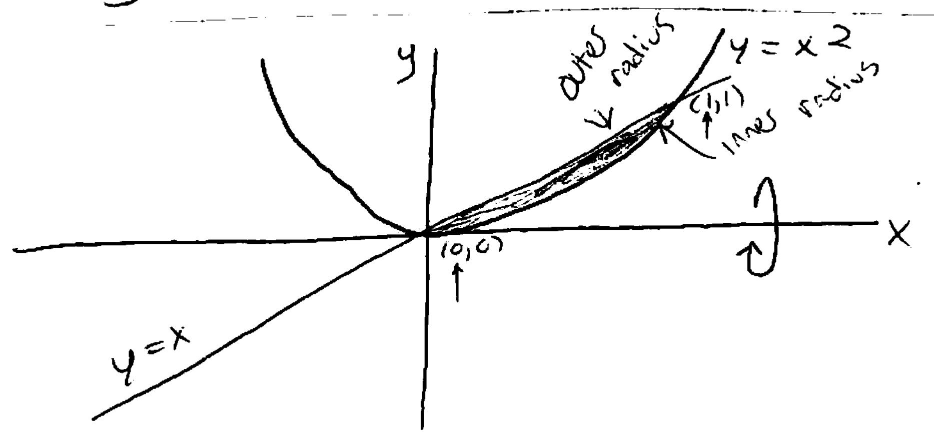
c. $y = 1$

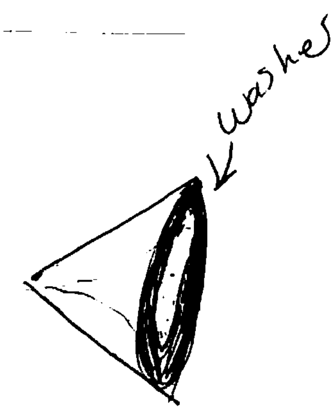
d. $y = -2$

e. $x = 1$

f. $x = -3$

a)



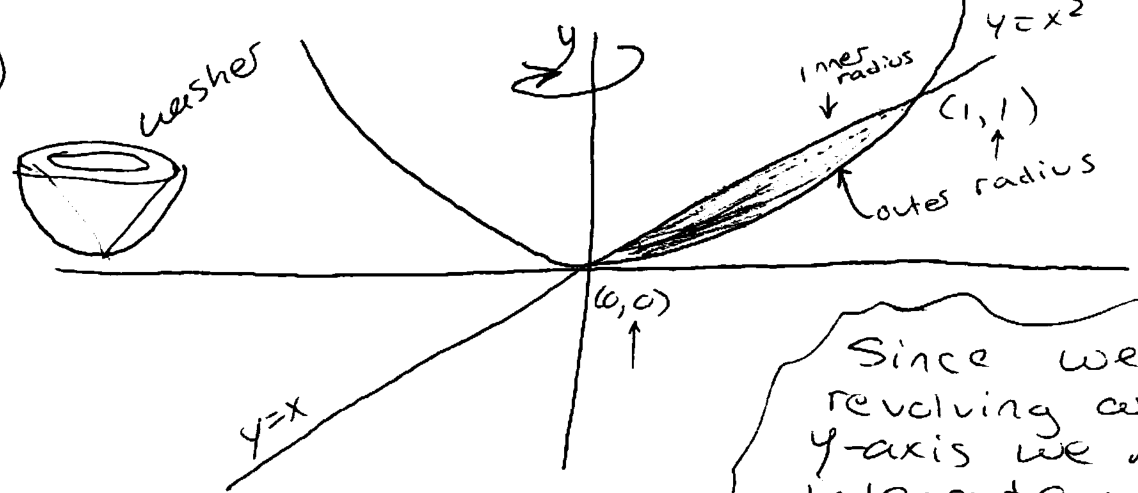


$x = x^2 \to 0 = x^2 - x = x(x-1)$ $\quad x = 0 \quad x = 1$

$$\pi \int_0^1 x^2 - (x^2)^2 \, dx = \pi \int_0^1 x^2 - x^4 \, dx = \pi \left[\tfrac{1}{3}x^3 - \tfrac{1}{5}x^5 \right] \Big|_0^1$$

$$\pi \left[\tfrac{1}{3} - \tfrac{1}{5} \right] = \boxed{\frac{2\pi}{15}}$$

b)



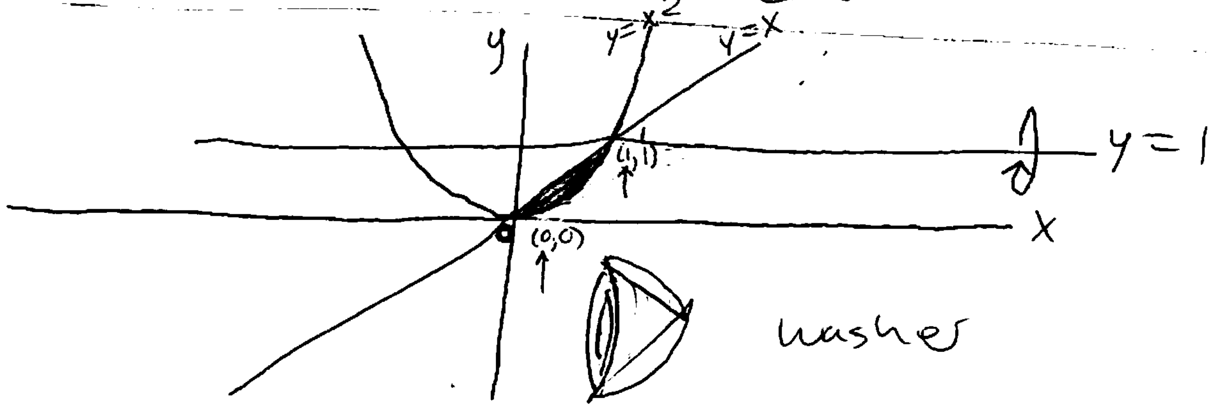
Since we are revolving around the y-axis we need to integrate with respect to y

$y=x^2$ in the 1st quadrant is equivalent to $x=\sqrt{y}$

$$\pi\int_0^1 (\sqrt{y})^2-(y)^2\,dy = \pi\int_0^1 y-y^2\,dy =$$

$$\pi\left[\tfrac{1}{2}y^2-\tfrac{1}{3}y^3\right] = \pi\left[\tfrac{1}{2}-\tfrac{1}{3}\right] = \boxed{\frac{\pi}{6}}$$

c)

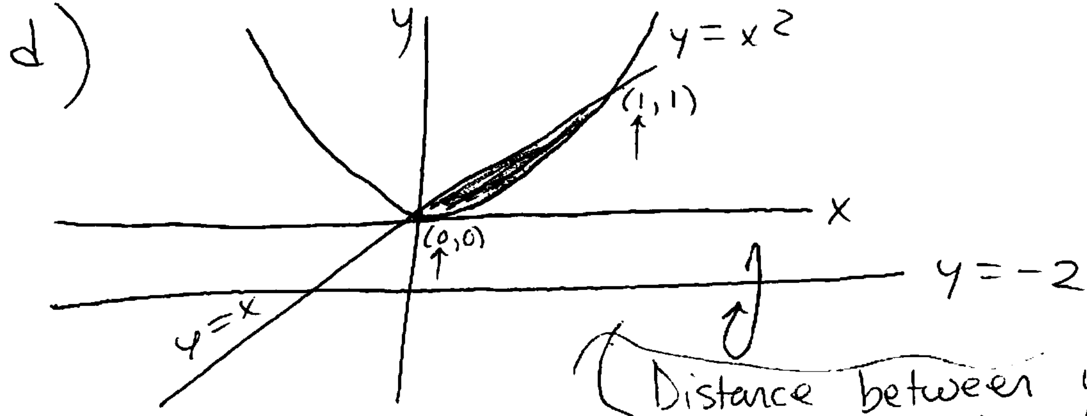


Note: distance between $x^2=y$ and $y=1$ is $(1-x^2)$ & Distance between $y=x$ and $y=1$ is $(1-x)$

$$\pi\int_0^1 (1-x^2)^2-(1-x)^2\,dx = \pi\int_0^1 1-2x^2+x^4-1+2x-x^2\,dx$$

$$= \pi\left[-x^3+\tfrac{1}{5}x^5+x^2\right]\Big|_0^1 = \pi\left[-1+\tfrac{1}{5}+1\right] = \boxed{\frac{\pi}{5}}$$

d)



Washer

Distance between $y=x$ and $y=-2$ is $2+x$ & distance between $y=x^2$ and $y=-2$ is x^2+2 (Note we want the distance to increase)

$$\pi\int_0^1 (2+x)^2-(2+x^2)^2\,dx$$
$$=\pi\int_0^1 4+4x+x^2-4-4x^2-x^4\,dx$$
$$=\pi\int_0^1 4x-3x^2-x^4\,dx$$
$$=\pi\left[2x^2-x^3-\tfrac{1}{5}x^5\right]_0^1$$
$$=\pi\left[2-1-\tfrac{1}{5}\right]=\boxed{\frac{4\pi}{5}}$$

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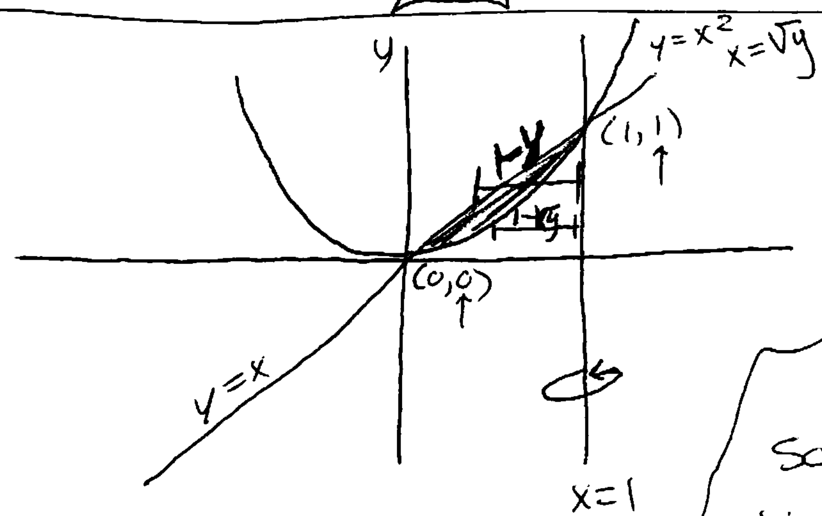
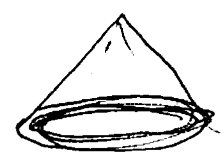
e)



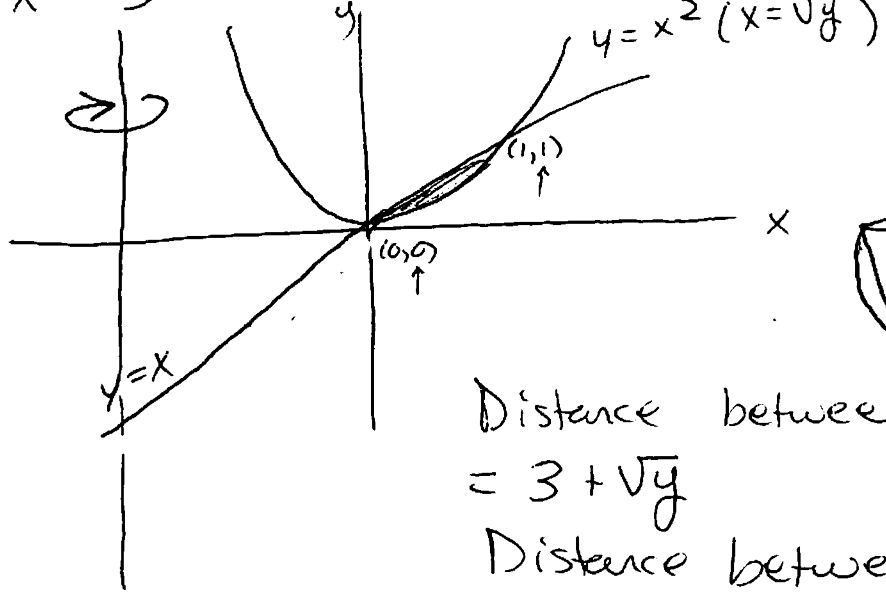


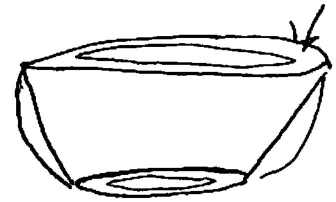
washer

Integrating about something // to the y-axis means our limits of integration, integrand must all be in terms of y

$$\pi\int_0^1 (1-y)^2-(1-\sqrt{y})^2\,dy$$
$$=\pi\int_0^1 1-2y+y^2-1+2\sqrt{y}-y\,dy$$
$$=\pi\int_0^1 -3y+y^2+2y^{1/2}\,dy$$
$$=\pi\left[-\tfrac{3}{2}y^2+\tfrac{1}{3}y^3+\tfrac{4}{3}y^{3/2}\right]_0^1=\pi\left[-\tfrac{3}{2}+\tfrac{1}{3}+\tfrac{4}{3}\right]=\pi\left[\tfrac{-9}{6}+\tfrac{10}{6}\right]=\boxed{\frac{\pi}{6}}$$

f. $x=-3$





Distance between $x=\sqrt{y}$ & $x=-3$
$= 3+\sqrt{y}$

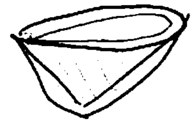
Distance between $x=y$ & $x=-3$
$= 3+y$

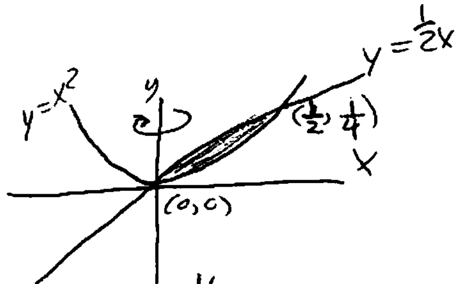
$$\pi\int_0^1 (3+\sqrt{y})^2-(3+y)^2\,dy = \pi\int_0^1 (9+6\sqrt{y}+y)-\cancel{9}-6y-y^2\,dy$$

$$=\pi\int_0^1 6y^{1/2}-5y-y^2\,dy = \pi\left(\frac{12}{3}y^{3/2}-\frac{5}{2}y^2-\frac{1}{3}y^3\right)\Big|_0^1$$

$$=\pi\left(\frac{12}{3}-\frac{5}{2}-\frac{1}{3}\right)=\pi\left(\frac{24}{6}-\frac{15}{6}-\frac{2}{6}\right)=\boxed{\frac{\pi\,17}{6}}$$

Note: If we were discussing the region bounded by $y=x^2$, $y=\frac{1}{2}x$ b & e would not be equal.





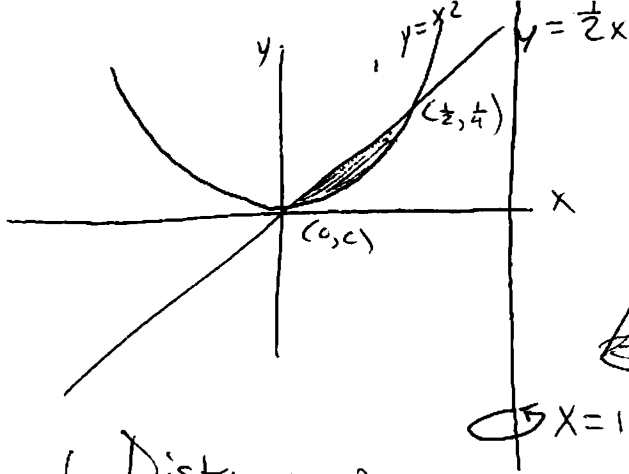
Observe:

$y=x^2$, $y=\frac{1}{2}x$ about the y-axis
$x=\sqrt{y}$, $x=2y$

$$\pi\int_0^{\frac14}(\sqrt{y})^2-(2y)^2\,dy=\pi\int_0^{1/4} y-4y^2\,dy=\pi\left[\frac{1}{2}y^2-\frac{4}{3}y^3\right]_0^{1/4}$$

$$=\pi\left[\frac{1}{32}-\frac{4}{3\cdot 64}\right]=\pi\left[\frac{1}{32}-\frac{1}{3\cdot 16}\right]=\pi\left[\frac{48-32}{32\cdot 48}\right]$$

$$=\pi\left[\frac{16}{32\cdot 48}\right]=\pi\left[\frac{1}{2\cdot 48}\right]=\pi\left[\frac{1}{96}\right]$$



Compare to:

$y=x^2,\quad y=\frac{1}{2}x$

about $x=1$

$x=\sqrt{y}\qquad x=2y$

$$\pi\int_0^{1/4}(1-2y)^2-(1-\sqrt{y})^2\,dy$$

$$=\pi\int_0^{1/4}1-4y+4y^2-1+2\sqrt{y}-y\,dy$$

$$=\pi\int_0^{1/4}-5y+4y^2+2y^{1/2}\,dy$$

$$=\pi\left[-\frac{5}{2}y^2+\frac{4}{3}y^3+\frac{4}{3}y^{3/2}\right]\Bigg|_0^{1/4}$$

$$=\pi\left[\frac{-5}{32}+\frac{1}{3\cdot 16}+\frac{4}{24}\right]=\boxed{\frac{\pi}{32}}$$

Distance from $x=2y$ to $x=1$ is $(1-2y)$. Distance from $x=\sqrt{y}$ to $x=1$ is $(1-\sqrt{y})$