Work Emptying A Tank

In class we derived the formula

top of liquid

$$\int_a^b \rho\, A(y)\,(h-y)\,dy$$

bottom of liquid

weight density

area of a horizontal cross section

how far a slab is lifted!

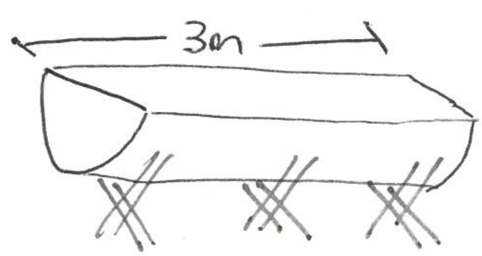
☆ This set up is if the x-axis is at the bottom of the tank. If we move the x-axis we will need to adjust.

Examples from Calculus Books

by Stewart, Briggs/Cochran/Gillett & Larson/Hostetler/Edwards

Ex 1 A water trough has a semicircular vertical cross section with a radius of .25m & a length of 3m

a) How much work is required to pump the water out of the top of the trough when it is full?



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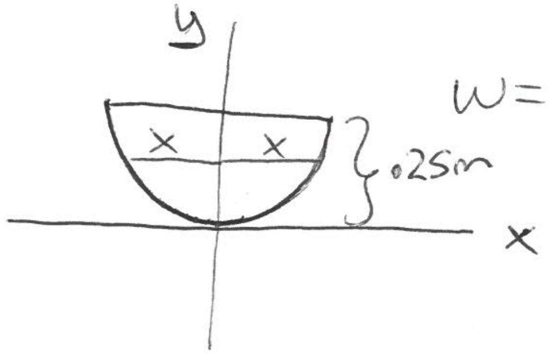
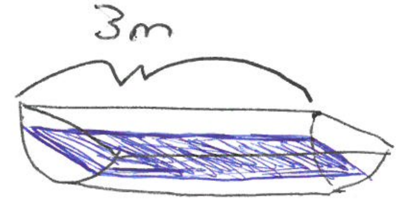


$w = 2x$

$A(y) = lw = 3w = 3(2x)$



★ Horizontal cross sections are rectangles the length is fixed & the width changes

If the x-axis is at the bottom of the tank & the y-axis is along a line of symmetry if we have one, then in this case the circle is not centered at the origin. It is centered at $(0, .25)$

Equation of a circle: $(x-h)^2 + (y-k)^2 = r^2$

$$(x-0)^2 + (y-.25)^2 = .25^2$$

Solve for x: $x = \sqrt{.25^2 - (y-.25)^2}$

$$W = \int_a^b \rho A(y)(h-y)\,dy$$

$$\boxed{W = \int_0^{.25} \underbrace{\underset{\text{density of water}}{1000}\,\underset{\text{gravity}}{(9.8)}}_{\rho} \underbrace{\left[3\left(2\sqrt{.25^2-(y-.25)^2}\right)\right]}_{A(y)} (\underset{\text{top of tank}}{.25}-y)\,dy \text{ J}}$$

3

FYI! If the x-axis is at the top of the tank, then the circle is centered at the origin & we get

$$W = \int_{-.25}^{0} 1000(9.8)\left[3\left(2\sqrt{.25^2-y^2}\right)\right](-y)\,dy \text{ J}$$

b) If the length is doubled, is the required work doubled? Explain.

If the length is doubled then $A(y) = lw$

$$= 3 \cdot 2\, w = (3 \cdot 2)\, 2\sqrt{.25^2-(y-.25)^2}$$

Plugging into the answer from page 2:

$$W = \int_0^{.25} 1000(9.8)\left[3 \cdot 2\left(2\sqrt{.25^2-(y-.25^2)}\right)\right](.25-y)\,dy \text{ J}$$

$$= 2\int_0^{.25} 1000(9.8)\left[3\left(2\sqrt{.25^2-(y-.25^2)}\right)\right](.25-y)\,dy \text{ J}$$

Yes! This should also make sense if we think about lifting a horizontal slab that is twice as long.

4| If the radius is doubled, is the required work doubled? Explain

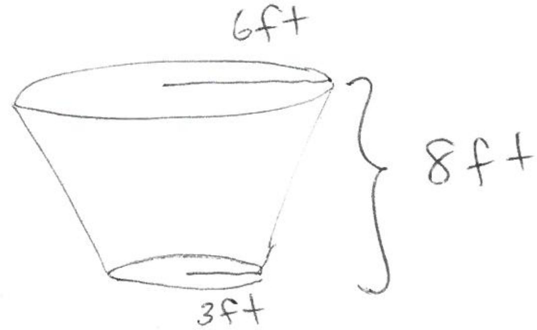
We'd get: $W = \int_0^{.5} 1000(9.8)\left[3\left(2\sqrt{.5^2-(y-.5)^2}\right)\right](.5-y)\,dy$

= 8 times our first answer!

No! If the radius is doubled, then we have to lift slabs further. It is not the same as emptying two identical tanks.

$\boxed{\text{Ex 2}}$ Find the work required to pump all of the water to the top of the tank, if the water starts 2ft below the top. Hint: water weighs 62.4 lb/ft^3

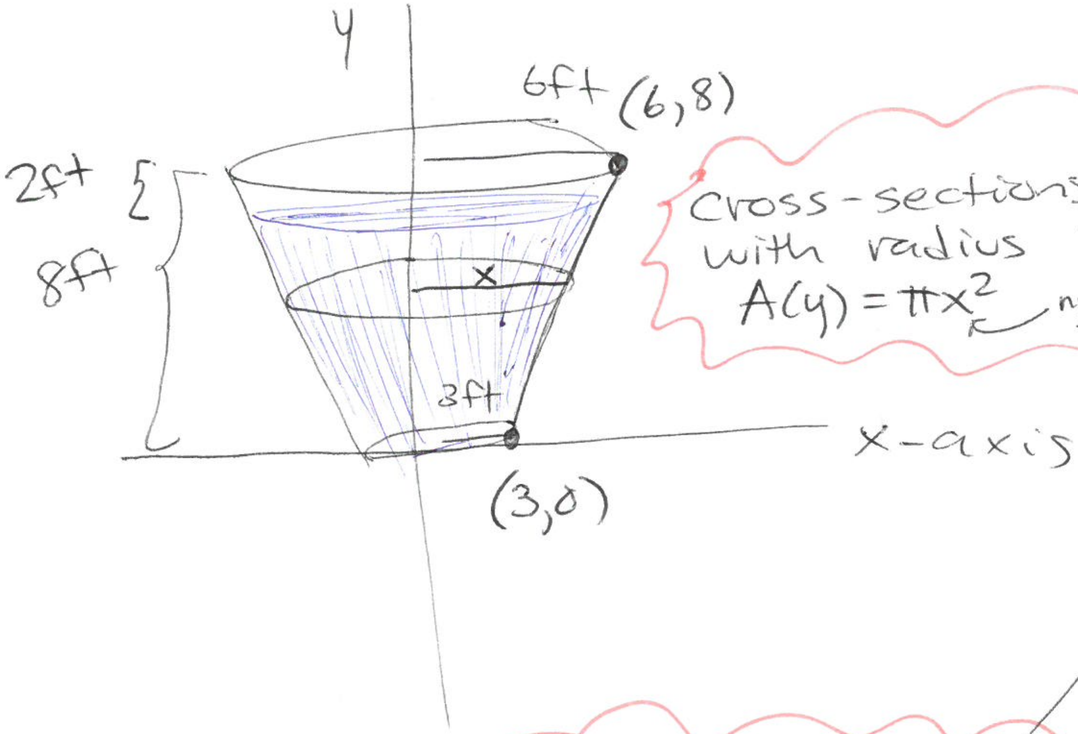
Given:



Frustum of a cone

probably not a word you hear all the time :)

5



Cross-sections are circles with radius x.

$A(y) = \pi x^2$ ← need to get in terms of y

To find x in terms of y, I am going to find the line & solve equation of the line to find x.

$$m = \frac{\Delta y}{\Delta x} = \frac{8-0}{6-3} = \frac{8}{3}$$

$$y = mx + b \rightarrow y = \frac{8}{3}x + b$$

(3,0) is on our line so

$$0 = \frac{8}{3} \cdot 3 + b \rightarrow b = -8$$

$$y = \frac{8}{3}x - 8 \quad \text{so} \quad y + 8 = \frac{8}{3}x$$

$$\boxed{x = \frac{3}{8}y + 3}$$

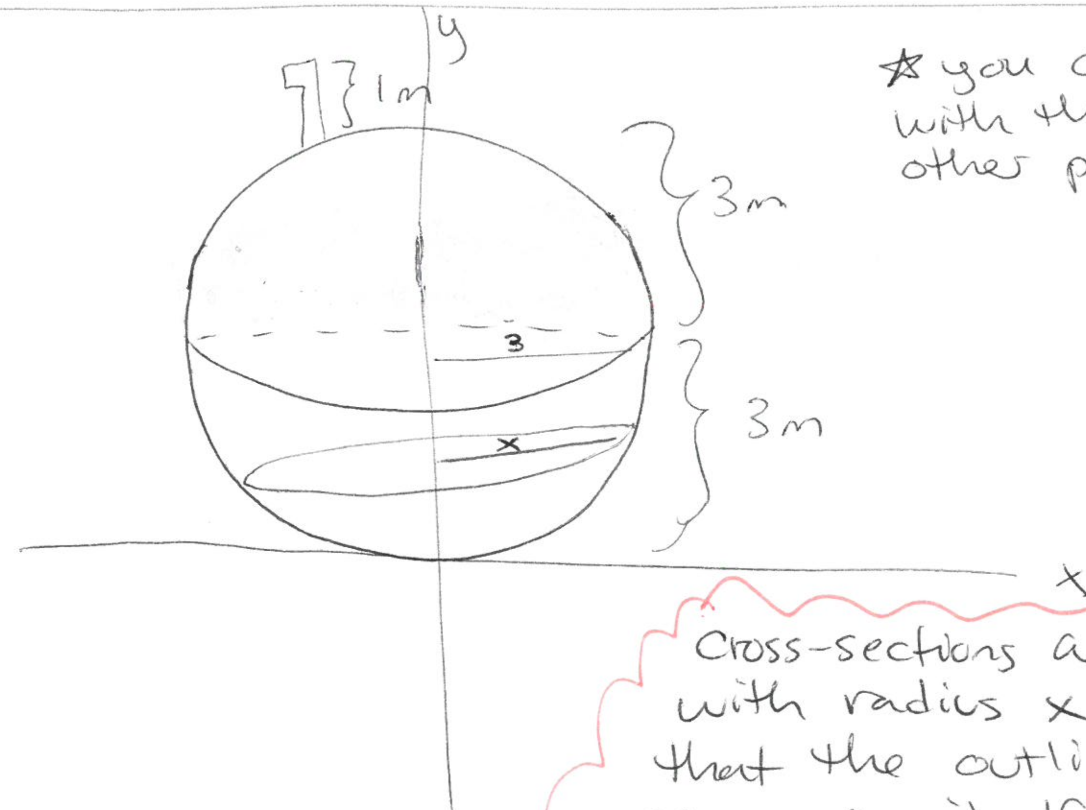
water ends 2ft below the top of the tank

$$W = \int_0^6 \underset{\uparrow \rho}{62.4} \underbrace{\left[\pi\left(\tfrac{3}{8}y+3\right)^2\right]}_{A(y)} \underset{\uparrow h}{(8-y)}\,dy \text{ ft-lb}$$

6| A tank is full of water. Find the work required to pump all of the water out of the spout

$\boxed{\text{Ex3}}$

Given:



☆ you can do this with the x-axis other places

Cross-sections are circles with radius x. Notice that the outline of the sphere is itself a circle for our set up it is centered at (0,3) so

$$(x-0)^2+(y-3)^2=3^2$$

$$x=\sqrt{9-(y-3)^2}$$

$$A(y)=\pi x^2=\pi(9-(y-3)^2)$$

$$W=\int_0^6 1000(9.8)\left[\pi(9-(y-3)^2)\right](7-y)\,dy \text{ J}$$

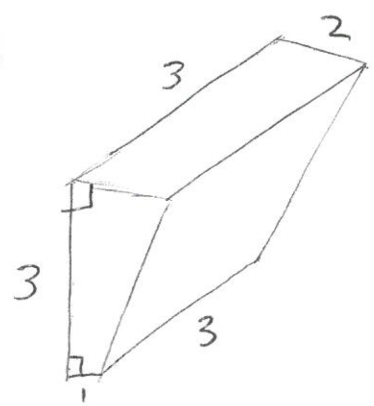
6+1 spout

7| Previous problem with the x-axis at the center of the sphere!

$$W = \int_{-3}^{3} 1000(9.8)\pi(9-y^2)(4-y)\,dy \text{ J}$$

Ex 4 The fuel tank on a large truck has trapezoidal cross sections with dimensions in ft as shown below. Assume the engine is approximately 2ft above the fuel tank & that diesel fuel weighs 55.6 lb/ft^3. Find the work done by the fuel pump raising a full tank of fuel to the level of the engine

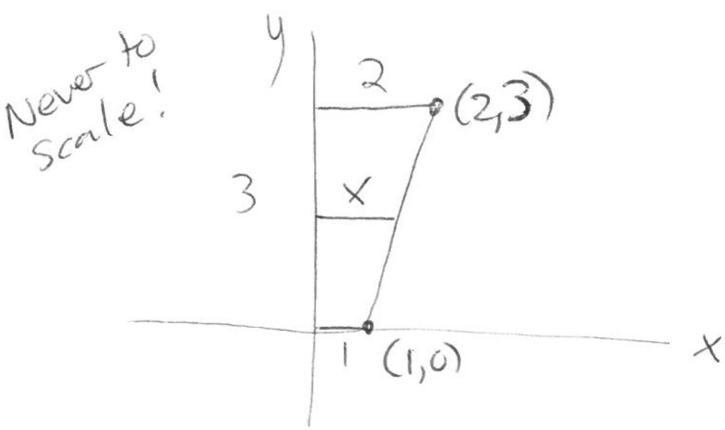
Given:



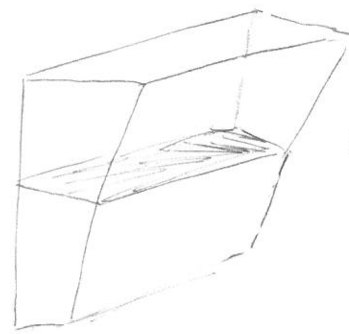
Please note! this problem clearly is delightful, but I regret to inform you that fuel tanks of large trucks seldom look trapezoidal. :(

8)

In 2D



In 3D



$A(y) = lw$

$= 3w$

$= 3x$

Like $\boxed{\text{Ex 2}}$, I'll find the equation of the line & solve for x $\quad m = \frac{\Delta y}{\Delta x} = \frac{3-0}{2-1} = 3$

$y = mx + b \rightarrow y = 3x + b$ $\quad (1,0)$ is on our line

so $0 = 3 \cdot 1 + b \rightarrow b = -3$ $\quad y = 3x - 3$

$\frac{y+3}{3} = x \qquad x = \frac{y}{3} + 1$

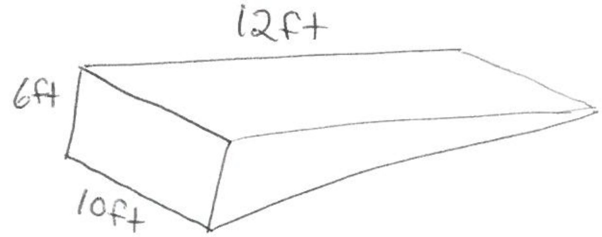
$$\boxed{W = \int_0^3 55.6\left(3\left(\frac{y}{3}+1\right)\right)(5-y)\,dy \text{ ft-lbs}}$$

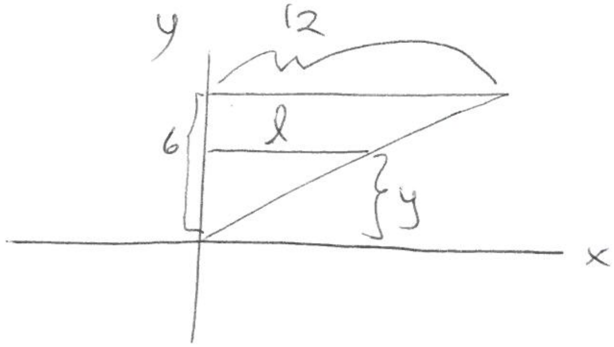
engine is 2ft above fuel tank

Ex 6

A tank is full of water. Find the work required to pump the water out of the top of the tank

Given:





★ Horizontal cross sections will be rectangle of width 10ft & changing length. I drew the triangle since we need to determine the equation of changing length

Similar triangles: $\frac{l}{y} = \frac{12}{6}$

$l = 2y$ Finding the eqn of the line like the last ex would work too!

$$\boxed{W = \int_0^6 62.4\left(\underbrace{10}_{w}(\underbrace{2y}_{l})\right)(6-y)\,dy \text{ ft-lbs}}$$