Trickier Final Exam Review Problems

- Suitable as a studying supplement for 241-040 & 241-050 students

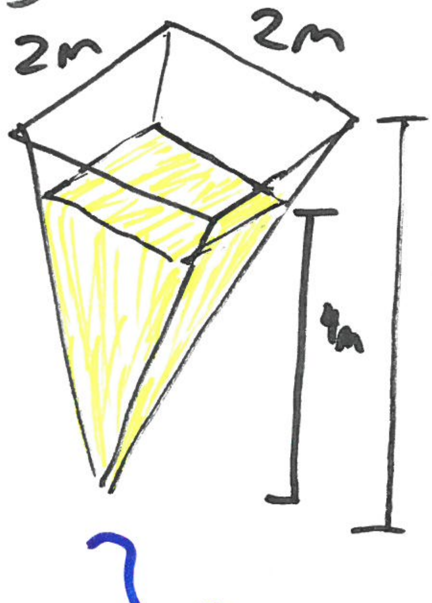
Not every problem on your final exam will be at this level (obviously :)).

Examples from Calculus Early Transcendentals by Sullivan & Miranda & Calculus by Edwards & Penney & made up

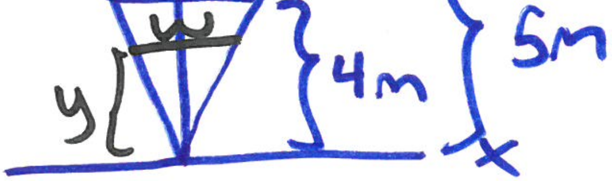
Ex 1 A container in the shape of an inverted pyramid with a square base of 2 m by 2 m & a height of 5 m is filled to a depth of 4 m with corn slurry. How much work is required to pump all the slurry over the top of the container?

The density of corn slurry is $17.9\ kg/m^3$

☆ Not that hard -- I would provide a picture



$A(y) = w^2$ ← square



$\frac{w}{y} = \frac{2}{5}$

$$W = \int_0^4 17.9(9.8)\left[\tfrac{2}{5}y\right]^2(5-y)\,dy \text{ J}$$

P3 | $\boxed{\text{Ex 2}}$ Find the unstretched length of a spring if the work required to stretch the spring from 1 to 1.4m is 1/2 the work required to stretch it from 1.2 to 1.8 m.

$$\underbrace{W = \int_a^b kx\,dx \qquad F = kx}_{\text{for springs}}$$

L = natural length of the spring

$$\int_{L-1}^{L-1.4} \cancel{k}x\,dx = \frac{1}{2}\int_{L-1.2}^{L-1.8} \cancel{k}x\,dx$$

$$\frac{1}{2}x^2\Big|_{L-1}^{L-1.4} = \frac{1}{4}x^2\Big|_{L-1.2}^{L-1.8}$$

$$2x^2\Big|_{L-1}^{L-1.4} = x^2\Big|_{L-1.2}^{L-1.8}$$

$$2\left[\cancel{L^2} - 2.8L + 1.4^2 - \cancel{L^2} + 2L - 1\right] = L^2 - 3.6L + 1.8^2 - \cancel{L^2} + 2.4L - 1.2^2$$

$$2\left[-.8L + .96\right] = -1.2L + 1.8$$

P4

$-.8L + .96 = -.6L + .9$

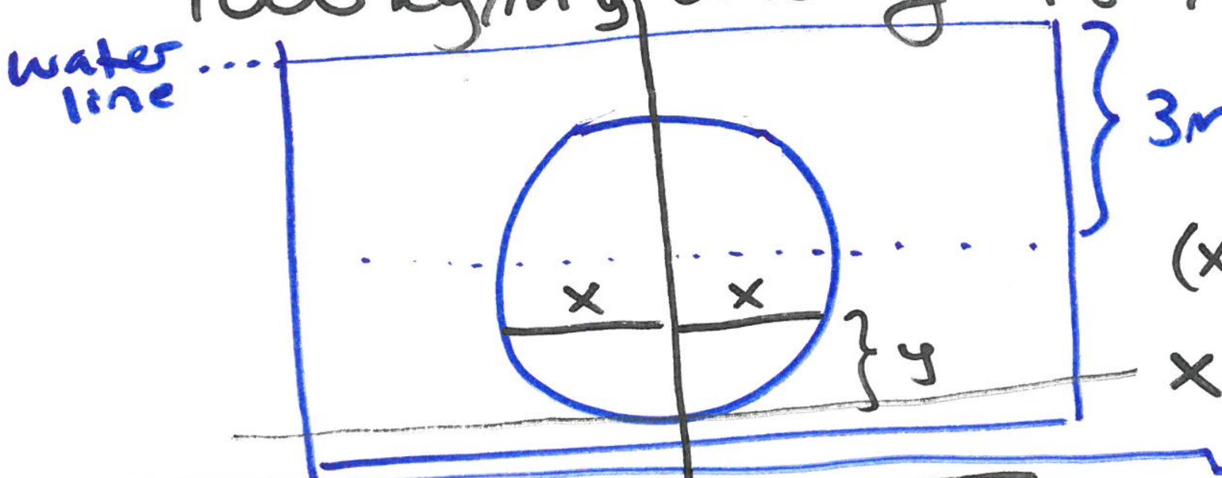
$-.2L = -.06$

$2L = .6$

$\boxed{L = .3\,m}$

* Probably would make the numbers whole numbers to make it easier.

Ex 3 A round window of radius 2m is built into the side of a large fresh water aquarium tank. If the center of the window is 3m below the water line, find the fluid force on the window. The density of fresh water is 1000 kg/m³, and g = 9.8 m/s²



$(x-0)^2 + (y-2)^2 = 4$

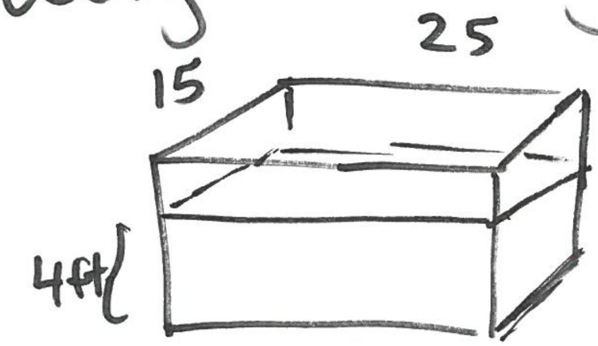
$x = \sqrt{4-(y-2)^2}$

$L = 2x$

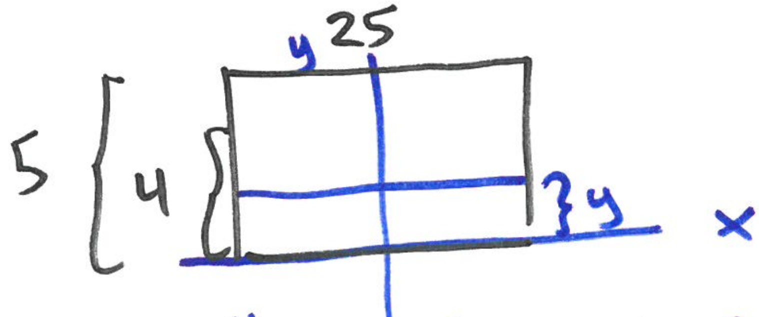
$$\boxed{F = \int_0^4 1000(9.8)\underbrace{2\sqrt{4-(y-2)^2}}_{L(y)}\underbrace{(5-y)}_{\text{depth}}\,dy \text{ N}}$$

P5

$\boxed{\text{Ex 4}}$ A motor can do 550 ft-lb of work per second. The motor is used to pump the water out of a swimming pool in the shape of a rectangular parallelepiped 5 ft deep, 25 ft long, & 15 ft wide. How long does it take for the pool to empty if it is filled to a depth of 4 ft? Use $\rho = 62.4 \text{ lb/ft}^3$ for the weight density of water.



$$A(y) = 15 \cdot 25$$
$$= 1250$$



$$W = \int_0^4 62.4(1250)(5-y)\,dy \text{ ft-lb}$$

P6

$$W = 62.4(1250)\left(5y - \tfrac{1}{2}y^2\right)\Big|_0^4$$
$$= 62.4(1250)[20 - 8]$$
$$= 62.4(1250)(12) \text{ ft-lb}$$

$$\text{Time} = \frac{62.4(1250)(12)}{550} \text{ seconds}$$

$\boxed{\text{Ex 5}}$ too hard $\int x \sin^{-1} x \, dx$

Hint: 1st do integration by parts

$$\int u\,dV = uv - \int v\,du$$

LIATE inverse trig

$u = \sin^{-1} x \qquad V = \tfrac{1}{2}x^2$

$du = \frac{dx}{\sqrt{1-x^2}} \qquad dV = x\,dx$

$uv - \int v\,du$

$$\sin^{-1} x \, \tfrac{1}{2}x^2 - \int \tfrac{1}{2}x^2 \frac{1}{\sqrt{1-x^2}}\,dx$$

P7

We need to integrate

$$\int \frac{1}{2} x^2 \frac{1}{\sqrt{1-x^2}}\, dx$$

Trig substitution!

$x = 1\sin\theta$

$dx = \cos\theta\, d\theta$

$$= \int \frac{1}{2} \sin^2\theta \frac{1}{\sqrt{1-\sin^2\theta}} \cos\theta\, d\theta$$

$$= \int \frac{1}{2} \sin^2\theta \frac{1}{\cancel{\sqrt{\cos^2\theta}}} \cancel{\cos\theta}\, d\theta$$

$\frac{1}{2}$ angle identity: (would be provided on final)

$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$

$$= \int \frac{1}{2} \cdot \frac{1}{2} (1-\cos 2\theta)\, d\theta$$

$$= \frac{1}{4}\left[\theta - \frac{1}{2}\sin 2\theta\right] + C$$

P8

$\frac{1}{4}\left[\theta - \frac{1}{2}\sin 2\theta\right] + C$

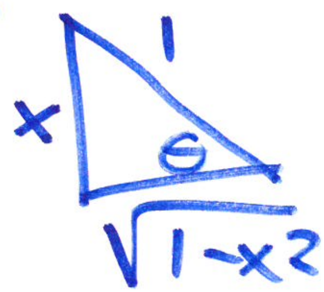
This is getting to be a bit much!

$\left.\sin 2A = 2\sin A\cos A\right\}$ would be given

$= \frac{1}{4}\left[\theta - \cancel{\tfrac{1}{2}}(\cancel{2}\sin\theta\cos\theta)\right] + C$

$x = 1\sin\theta = \frac{opp}{hyp}$

$\theta = \sin^{-1}x$



$= \frac{1}{4}\left[\sin^{-1}x - \frac{x}{1}\frac{\sqrt{1-x^2}}{1}\right] + C$

Final ans:

$$\boxed{\sin^{-1}x\,\frac{1}{2}x^2 - \frac{1}{4}\sin^{-1}x + \frac{1}{4}x\sqrt{1-x^2} + C}$$

$\boxed{\text{Ex 6}}$ $\int x \tan^{-1}x \, dx$

Hint: 1st do integration by parts

LIATE

$u = \tan^{-1}x \qquad v = \frac{1}{2}x^2$

$du = \frac{1}{1+x^2}dx \qquad dV = x\,dx$

$uv - \int v\,du$

$\tan^{-1}x\left(\frac{1}{2}x^2\right) - \int \frac{1}{2}x^2 \frac{1}{1+x^2}dx$

Trig sub!

$x = 1\tan\theta$

$dx = \sec^2\theta \, d\theta$

$\tan^{-1}x\left(\frac{1}{2}x^2\right) - \int \frac{1}{2}\tan^2\theta \frac{1}{\underbrace{1+\tan^2\theta}_{\cancel{\sec^2\theta}}} \cancel{\sec^2\theta}\, d\theta$

$= \tan^{-1}x\left(\frac{1}{2}x^2\right) - \int \frac{1}{2}\tan^2\theta \, d\theta$

P 10

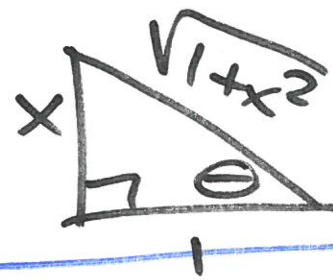
$$\tan^{-1}x\left(\tfrac{1}{2}x^2\right) - \int \tfrac{1}{2}\tan^2\theta \, d\theta$$

$$= \tan^{-1}x\left(\tfrac{1}{2}x^2\right) - \int \tfrac{1}{2}\left(\sec^2\theta - 1\right) d\theta$$

$$= \tan^{-1}x\left(\tfrac{1}{2}x^2\right) - \tfrac{1}{2}\left[\tan\theta - \theta\right] + C$$

$x = 1\tan\theta$

$\frac{x}{1} = \tan\theta = \frac{\text{opp}}{\text{adj}}$



$$= \boxed{\tan^{-1}x\left(\tfrac{1}{2}x^2\right) - \tfrac{1}{2}\left[x - \tan^{-1}x\right] + C}$$

I wasn't paying attention, no $x = \tan\theta$, no need for a triangle for this one.

$\boxed{\text{Ex 7}}$ $\int e^x \sqrt{25 - e^{2x}}\, dx$

Hint: 1st do a u-sub

$25 - (e^x)^2$

$u = e^x$
$du = e^x\, dx$

$= \int \sqrt{25 - u^2}\, du$

$u = 5\sin\theta$
$du = 5\cos\theta\, d\theta$

$= \int \sqrt{25 - 25\sin^2\theta}\; 5\cos\theta\, d\theta$

$= \int \sqrt{25\cos^2\theta}\; 5\cos\theta\, d\theta$

$= \int 25\cos^2\theta\, d\theta$

$= \int 25\, \frac{1}{2}(1 + \cos 2\theta)\, d\theta$

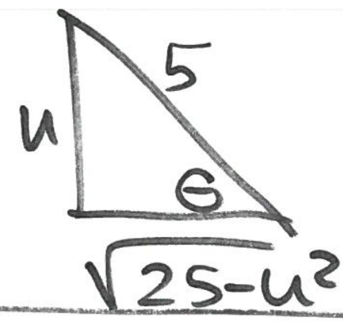
$= 25/2 \left[\theta + \frac{1}{2}\sin 2\theta\right] + C$

Ex 8 $\frac{25}{2}\left[\theta + \frac{1}{\cancel{2}}\underbrace{\cancel{2}\sin\theta\cos\theta}_{\text{trig identity}}\right] + C$

$$= \frac{25}{2}\left[\sin^{-1}\left(\frac{u}{5}\right) + \frac{u}{5}\,\frac{\sqrt{25-u^2}}{5}\right] + C$$

$u = 5\sin\theta$

$\frac{u}{5} = \sin\theta = \frac{opp}{hyp}$



$$\boxed{= \frac{25}{2}\left[\sin^{-1}\left(\frac{e^x}{5}\right) + \frac{e^x}{5}\,\frac{\sqrt{25-e^{2x}}}{5}\right] + C}$$

Ex 9 too hard $\int \frac{\sec^2 x \; dx}{\sqrt{\tan^2 x - 6\tan x + 8}}$

Hint: u-sub, complete the sq, u-sub, trig sub :-

This is probably too much but let's do it anyway

P13

$u = \tan x$

$du = \sec^2 x \, dx$

$$\int \frac{du}{\sqrt{u^2 - 6u + 8}}$$

☆ this wouldn't be too bad if it started here

$$= \int \frac{du}{\sqrt{\underbrace{(u-3)^2}_{u^2-6u+9} - 1}}$$

$v = u - 3$

$dv = du$

☆ Trig sub!

$$= \int \frac{dv}{\sqrt{v^2 - 1}}$$

$v = 1\sec\theta$

$dv = \sec\theta \tan\theta \, d\theta$

$$= \int \frac{\sec\theta \cancel{\tan\theta} \, d\theta}{\underbrace{\sqrt{\sec^2\theta - 1}}_{\cancel{\tan\theta}}}$$

$$= \int \sec\theta \, d\theta$$

☆ you don't have to know how to integrate $\sec\theta$ or $\sec^3\theta$

$$= \int \sec\theta \cdot \frac{\sec\theta + \tan\theta}{\sec\theta + \tan\theta} \, d\theta$$

P14

$$\int \frac{\sec^2\theta + \sec\theta\tan\theta}{\sec\theta + \tan\theta}\, d\theta$$

Another substitution:

$w = \sec\theta + \tan\theta$

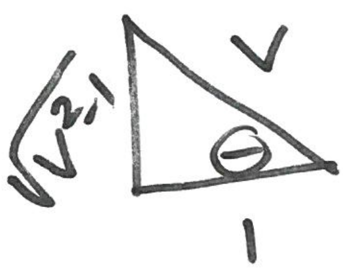
$dw = \sec\theta\tan\theta + \sec^2\theta \, d\theta$

$$\int \frac{1}{w}\, dw = \ln|w| + C$$

$$= \ln|\sec\theta + \tan\theta| + C$$

$V = 1\sec\theta$

$\frac{V}{1} = \sec\theta = \frac{1}{\cos\theta} = \frac{hyp}{adj}$



$$= \ln\left|v + \sqrt{v^2-1}\right| + C$$

$$= \ln\left|u-3 + \sqrt{(u-3)^2-1}\right| + C$$

$$= \ln\left|\tan x - 3 + \sqrt{(\tan x-3)^2 - 1}\right| + C$$

P15

$\boxed{\text{Ex 10}}$ $\int \frac{\sin\theta\cos\theta}{\cos^3\theta+9\cos\theta}\,d\theta$

Hint: u-sub, then partial fractions

$u=\cos\theta$

$du=-\sin\theta\,d\theta$

$$=\int \frac{-u\,du}{u^3+9u}=\int\frac{-u\,du}{u(u^2+9)}=$$

$$\int \frac{A}{u}+\frac{Bu+C}{u^2+9}\,du$$

$$A(u^2+9)+(Bu+C)u=-u$$

$$Au^2+9A+Bu^2+Cu=-u$$

$A+B=0$

$C=-1$

$9A=-1$

$A=-1/9 \quad B=1/9$

$$\int \frac{-1/9}{u}+\frac{1/9\,u}{u^2+9}-\frac{1}{u^2+9}\,du$$

P16

$$= -\frac{1}{9}\ln|u| + \underbrace{\frac{1}{18}\ln(u^2+9)}_{\text{did this in my head – otherwise } w=u^2+9,\ dw=2u\,du} - \frac{1}{3}\tan^{-1}\left(\frac{u}{3}\right) + C$$

did this in
my head –
otherwise
$w = u^2+9$
$dw = 2u\,du$

⋮

$$= \boxed{-\frac{1}{9}\ln|\cos\theta| + \frac{1}{18}\ln(\cos^2\theta+9) - \frac{1}{3}\tan^{-1}\left(\frac{\cos\theta}{3}\right) + C}$$

$\boxed{\text{Ex 11}}$

$$\int_0^\infty \frac{x}{(1+x)^3}\,dx$$

$$= \int_1^\infty \frac{u-1}{u^3}\,du =$$

$u = 1+x \rightarrow x = u-1$

$du = dx$

$u(0) = 1+0 = 1$

$u(\text{"}\infty\text{"}) = \text{"}\infty\text{"}$

P17

$$\lim_{t\to\infty} \int_1^t \frac{u}{u^3} - \frac{1}{u^3}\, du$$

$$= \lim_{t\to\infty} \int_1^t u^{-2} - u^{-3}\, du$$

$$= \lim_{t\to\infty} -u^{-1} + \frac{1}{2}u^{-2} \Big|_1^t$$

$$= \lim_{t\to\infty} -\frac{1}{t} + \frac{1}{2}\frac{1}{t^2} - \left[-\frac{1}{1} + \frac{1}{2}\right]$$

$= \boxed{1/2}$ finite so converges

☆ You could also integrate with partial fractions but that would be harder

$$\int \frac{x}{(1+x)^3} = \int \frac{A}{1+x} + \frac{B}{(1+x)^2} + \frac{C}{(1+x)^3}\, dx$$

$\vdots$

P18

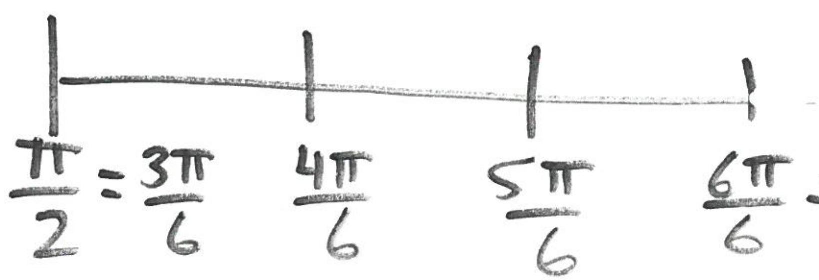
$\boxed{\text{Ex 12}}$

a) Approximate $\int_{\pi/2}^{\pi} \frac{\sin x}{x}\,dx$ $n=3$

with Trapezoidal

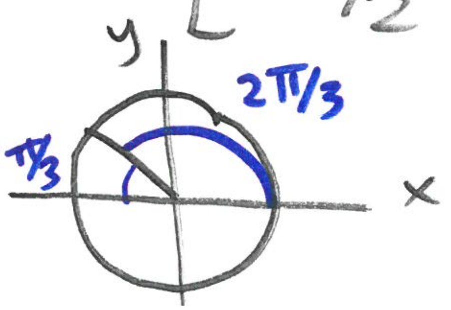
$$T_n = \frac{\Delta x}{2}\Big[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\Big]$$

$$\Delta x = \frac{b-a}{n} = \frac{\pi - \pi/2}{3} = \frac{\pi}{6}$$



$$T_3 = \frac{\pi/6}{2}\Big[f\left(\frac{\pi}{2}\right) + 2f\left(\frac{2\pi}{3}\right) + 2f\left(\frac{5\pi}{6}\right) + f(\pi)\Big]$$

$$= \frac{\pi}{12}\left[\frac{\overset{1}{\cancel{\sin\frac{\pi}{2}}}}{\pi/2} + 2\,\frac{\overset{\sqrt{3}/2}{\cancel{\sin\frac{2\pi}{3}}}}{\frac{2\pi}{3}} + 2\,\frac{\overset{1/2}{\cancel{\sin\frac{5\pi}{6}}}}{\frac{5\pi}{6}} + \overset{0}{\cancel{\frac{\sin\pi}{\pi}}}\right]$$



P19

$$\int_{\pi/2}^{\pi} \frac{\sin x}{x}\,dx \approx \frac{\pi}{12}\left[\frac{2}{\pi} + \frac{3\sqrt{3}}{2\pi} + \frac{6}{5\pi}\right]$$

b) Find $\int_{\pi/2}^{\pi} \frac{\sin x}{x}\,dx$ exactly using its Maclaurin series.

Hint: $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

(I'd either give you this or have you derive it)

$$\frac{\sin x}{x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}$$

$$\int_{\pi/2}^{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}\,dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!\,(2n+1)} \Bigg|_{\pi/2}^{\pi}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{(2n+1)!\,(2n+1)} - \sum_{n=0}^{\infty} \frac{(-1)^n (\pi/2)^{2n+1}}{(2n+1)!\,(2n+1)}$$

$\boxed{\text{Ex 13}}$ Given $y''+25y=0$

a) Show that the Bandary value Problem (BVP)

$y''+25y=0 \quad y(0)=0 \quad y(\pi)=0$

has infinitely many solutions

$r^2+25=0$

$y=c_1\cos 5x + c_2\sin 5x$

$y(0)=0=c_1\cos 0 + \cancel{c_2\sin 0}$

$c_1=0$

$y=c_2\sin 5x$

$y(\pi)=0=c_2\sin 5\pi = c_2\cdot 0=0$

Any value of c_2 works with these conditions

$y=c_2\sin 5x$

b) show that the conditions $y(0)=0$ & $y(\frac{\pi}{5})=10$ will result in no solution.

$y(0)=0 \rightarrow c_1=0 \quad y(\frac{\pi}{5})=c_2\sin\pi=0\neq 10$

$\boxed{\text{Ex 14}}$ An integral equation is an equation where an unknown function appears under the integral sign.

Solve the integral equation

$$y(x) = 2 + \int_3^x e^{y(t)}(t+1)\,dt$$

Fundamental thm:

$$\frac{dy}{dx} = 0 + e^{y(x)}(x+1)$$

$$\int \frac{dy}{e^y} = \int x+1\,dx$$

$$\int e^{-y}\,dy = \int x+1\,dx$$

$$-e^{-y} = \tfrac{1}{2}x^2 + x + C$$

$$e^{-y} = -\tfrac{1}{2}x^2 - x + C_1$$

$$-y = \ln\left(-\tfrac{1}{2}x^2 - x + C_1\right)$$

$$y = -\ln\left(-\tfrac{1}{2}x^2 - x + c_1\right)$$

$$y(x) = 2 + \int_3^x e^{y(t)}(t+1)\,dt$$

$$\rightarrow y(3) = 2 + \int_3^3 e^{y(t)}(t+1)\,dt$$

$$y(3) = 2$$

$$y(3) = 2 = -\ln\left(-\tfrac{1}{2}\cdot 9 - 3 + c_1\right)$$

$$e^{-2} = -\tfrac{9}{2} - 3 + c_1$$

$$c_1 = e^{-2} + \tfrac{9}{2} + 3$$

$$\boxed{y = -\ln\left(-\tfrac{1}{2}x^2 - x + e^{-2} + \tfrac{9}{2} + 3\right)}$$

Ex 15 Solve the integral equation as an explicit solution

$$y(x) = 4 + \int_0^x 2t\sqrt{y(t)}\,dt$$

P23

$$\frac{dy}{dx} = 0 + 2x\sqrt{y(x)}$$

$$\int \frac{dy}{\sqrt{y}} = \int 2x\,dx$$

$$\int y^{-1/2}\,dy = \int 2x\,dx$$

$$2\sqrt{y} = x^2 + C$$

$$\sqrt{y} = \frac{x^2}{2} + C_1$$

$$y = \left(\frac{x^2}{2} + C_1\right)^2$$

$$y(0) = 4 + \int_0^0 2t\sqrt{y(t)}\,dt$$

$$\rightarrow y(0) = 4$$

$$y(0) = 4 = \left(\frac{0^2}{2} + C_1\right)^2$$

looks like $C_1 = \pm 2$ but we had $\sqrt{y} = \frac{x^2}{2} + C_1$

P24

$y(0)=4$

$$\sqrt{4} = \frac{0^2}{2} + c_1 \rightarrow c_1 = 2$$

$$\boxed{y = \left(\frac{x^2}{2} + 2\right)^2}$$

Ex 16 Suppose that a nuclear accident was confined to a single room of a nuclear research laboratory but has left that room contaminated with polonium-210 which has a half-life of 140 days. If the initial contamination of the room is five times the amount safe for long-term human exposure, how long should the laboratory

P25 employees wait before entering the room to decontaminte it?

$\frac{1}{2}$ life problem → exponential decay

$\frac{dy}{dt} = ky$ $y = y_0 e^{kt}$

$y_0 = 5s$ ← safe amount

$y = 5se^{kt}$

$y(140) = \frac{5s}{2} = 5se^{140k}$

$\ln\frac{1}{2} = 140k$

$y = 5se^{\frac{t}{140}\ln(\frac{1}{2})} \stackrel{?}{=} s$

$e^{t/140 \ln(\frac{1}{2})} = \frac{1}{5}$

$\frac{t}{140}\ln(\frac{1}{2}) = \ln(\frac{1}{5})$

$t = \left(\ln(1/5) / \ln(\frac{1}{2})\right) 140 \approx 325.07$ days

Ex 17 A person promises to give you \$1000 on January 1 2020. Each day thereafter they will give you $\frac{9}{10}$ of the amount they gave on the previous day.

a) What is the total amount you will receive?

$$1000 + \frac{9}{10}1000 + \left(\frac{9}{10}\right)^2 1000 + \left(\frac{9}{10}\right)^3 1000 + \cdots$$

$$a + ar + ar^2 + \cdots$$

$a = 1000$ $r = \frac{9}{10}$ $\left|\frac{9}{10}\right| < 1$ Converges

$$\frac{a}{1-r} = \frac{1000}{1-\frac{9}{10}} = \frac{1000}{\frac{1}{10}} = \boxed{\$10{,}000}$$

Geometric Series

P 27

b) What is the first date on which the amount you receive is less than 1 cent?

$$\sum_{n=0}^{\infty} \left(\frac{9}{10}\right)^n 1000$$

$$\left(\frac{9}{10}\right)^n 1000 = \frac{1}{100}$$

1 cent in dollars

$$\left(\frac{9}{10}\right)^n \leq \frac{1}{100000} = .00001$$

$$n = 110$$

I don't care about what the actual date will be or factoring in that 2020 is a leap year. Part a) is more fun.

P 28

$\boxed{\text{Ex 18}}$ Solve for x:

$$\frac{x}{2+2x} = x+x^2+x^3+\cdots, \quad |x|<1$$

$a+ar+ar^2$

$a=x \quad r=x \quad |x|<1$

converges to

$$\frac{a}{1-r}$$

$$\frac{x}{2+2x} = \frac{x}{1-x}$$

$$x(1-x) = (2+2x)x$$

$$x-x^2 = 2x+2x^2$$

$$0 = 3x^2-x$$

$$0 = x(3x-1)$$

$$x=0, x=1/3$$

$\boxed{\text{Ex 19}}$ Is the following true or false? If false, give a counter example.

a) $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ converges

False $a_n = \frac{1}{n}$ $\lim_{n\to\infty} \frac{1}{n} = 0$

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges Harmonic series

∞ly more examples.

b) If $\sum_{n=1}^{\infty} a_n + b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ & $\sum_{n=1}^{\infty} b_n$ converges.

False $a_n = \frac{1}{n}$, $b_n = -\frac{1}{n}$

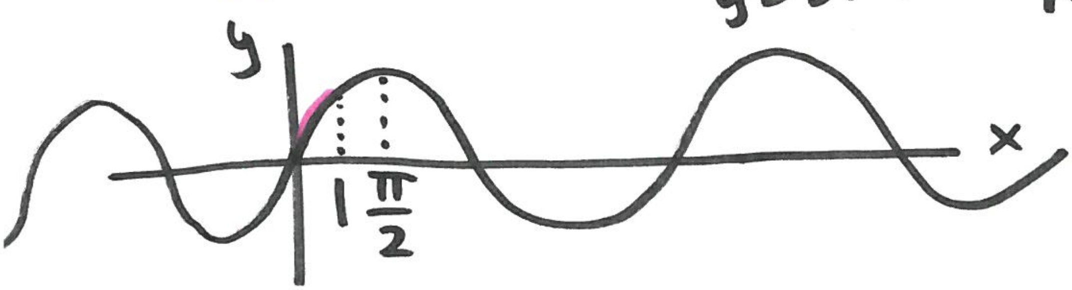
$\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n} = 0$ converges

Seperately they are Harmonic series.

P30

$\boxed{\text{Ex 20}}$ Use the LCT to determine if the following converges or diverges

$$\sum_{n=1}^{\infty} \sin\left(\tfrac{1}{n}\right)$$



Note! $\sin\left(\tfrac{1}{n}\right) > 0$

$\lim_{n\to\infty} \sin\left(\tfrac{1}{n}\right) = \sin 0 = 0$

but for finite n, $\sin\left(\tfrac{1}{n}\right) > 0$.

$$\lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} \quad \text{"}\frac{\sin 0}{0}\text{"} = \text{"}\frac{0}{0}\text{"}$$

$$= \lim_{x\to\infty} \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}} \overset{\lim_{x\to\infty}}{=} \frac{\cos\left(\frac{1}{x}\right)\cancel{\left(-\frac{1}{x^2}\right)}}{\cancel{\left(-1/x^2\right)}} = \cos 0 = 1 > 0$$

finite

$\rightarrow$ both series converge or diverge by LCT

$\sum \frac{1}{n}$ Harmonic diverges

so $\sum \sin\frac{1}{n}$ diverges by LCT

$\boxed{\text{Ex 21}}$ Show that the series

$$e^{-x}\cos x + e^{-2x}\cos 2x + e^{-3x}\cos 3x + \cdots$$

is absolutely convergent for all positive values of x.

$$\sum_{n=1}^{\infty} e^{-nx}\cos nx$$

never negative

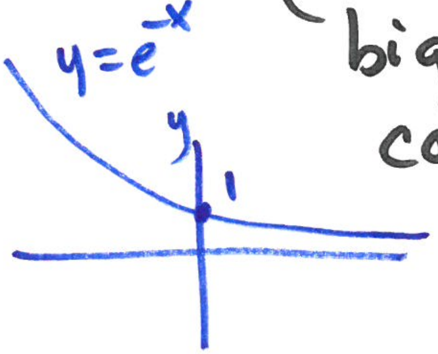
$$\sum_{n=1}^{\infty} |e^{-nx}\cos nx| = \sum_{n=1}^{\infty} e^{-nx}|\cos nx|$$

$$\leq \sum_{n=1}^{\infty} e^{-nx}\cdot 1 = \sum_{n=1}^{\infty} (e^{-x})^n = e^{-x} + (e^{-x})^2 + \ldots$$

$$a + ar + \ldots$$

biggest cosine can be

$r = e^{-x}$



if $x>0$, $e^{-x}<1$

P32

so $|e^{-x}| < 1$ for $x > 0$

$\therefore$ convergent Geometric series

so $\sum |e^{-nx} \cos nx|$ converges

by comparison test $\longrightarrow$

$\sum e^{-nx} \cos nx$ is absolutely convergent.

$\boxed{\text{Ex 22}}$

Determine if the series converges absolutely, conditionally or diverges

$1 + r\cos\theta + r^2 \cos 2\theta + r^3 \cos 3\theta + \dots$

$$\sum_{n=0}^{\infty} r^n \cos n\theta$$

$$\sum_{n=0}^{\infty} |r^n \cos n\theta| \le \sum_{n=0}^{\infty} |r^n|$$

Let $R = |r|$

$$= \sum_{n=0}^{\infty} R^n = R^0 + R^1 + R^2 + \cdots$$

$a + ar + ar^2$

Geometric

~~m~~ converges if $|R| < 1$

absolutely convergent by comparison test if $|r| < 1$

$\boxed{\text{Ex } 23}$ Find a power series representation for $\ln\left(\frac{1}{1-x}\right)$

$$f = \ln\left(\frac{1}{1-x}\right) = \ln 1 - \ln(1-x) = -\ln(1-x)$$

$$f' = \frac{-1}{1-x}(-1) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad |x| < 1$$

$$\ln\left(\frac{1}{1-x}\right) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} + C$$

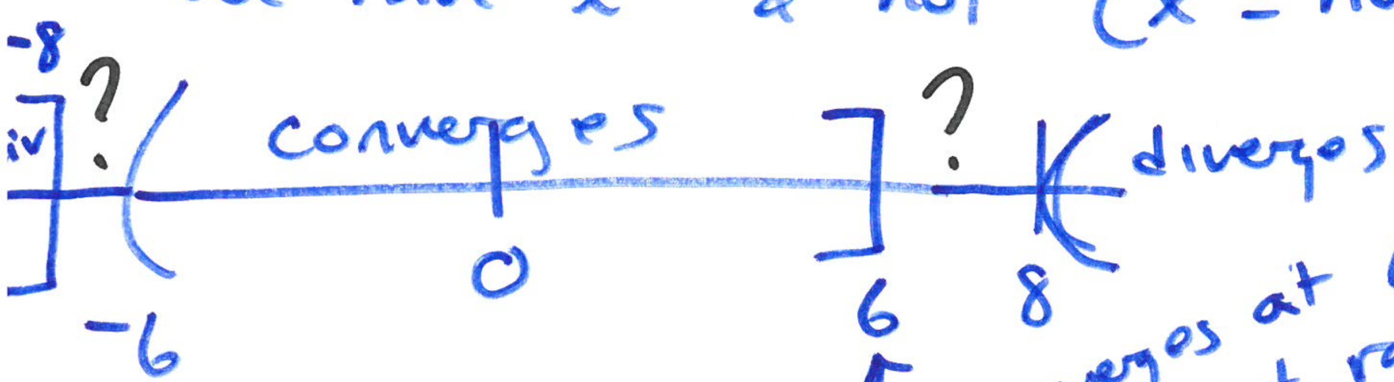
$$\ln\left(\frac{1}{1-0}\right) = \ln 1 = 0 = \sum_{n=0}^{\infty} \frac{0^{n+1}}{n+1} + C \rightarrow C = 0$$

$$\ln\left(\frac{1}{1-x}\right) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$$

$\boxed{\text{Ex 24}}$ If the series $\sum_{n=0}^{\infty} a_n x^n$ converges for $x=6$ & diverges for $x=-8$ what, if anything, can be said about the truth of the following statements?

a) the series converges for $x=2$

$\sum_{n=0}^{\infty} a_n x^n$ is a power series centered at 0 since we have x^n & not $(x \pm \text{number})^n$



converges at 6 so smallest radius of convergence is 6.

P35 | $x=2$ converges true

b) The series diverges for $x=7$

Unknown what happens at $x=7$, false

c) The series is absolutely convergent for $x=6$

False, Ratio test tells us absolute convergence ~~[illegible]~~ but R might be 6. At the endpoints we don't know

$$\sum_{n=1}^{\infty} \frac{(-x)^n}{6^n n}$$ Ratio test $|x|<6$ (try it yourself)

$x=6$ $\sum_{n=1}^{\infty} \frac{(-6)^n}{6^n n} = \sum \frac{(-1)^n}{n}$ which converges by AST but is not absolutely convergent.

d) The series converges for $x=-6$

False, unknown

e) the series diverges for $x=10$

True

f) The series is ab. conv. for $x=4$ true