Estimating Sums

* If we can show that a series converges using either the Alternating Series test or the Integral Test, we can estimate its sum

Examples mostly from Calculus for Scientists & Engineers by Briggs, Cochran, & Gillett

Recall: If $f(n)=a_n$ for $n=1,2,\cdots$ & $f(x)$ is positive & decreasing for $x \geq 1$ then

$$\sum_{n=1}^{\infty} a_n \quad \& \quad \int_1^{\infty} f(x)\,dx$$

either both converge or diverge

If they converge we can estimate the sum $\sum_{n=1}^{\infty} a_n$ using formulas we derived in class.

If $S=\sum_{n=1}^{\infty} a_n$ & $\underbrace{S_n}_{n^{th}\text{ partial sum}} = \sum_{i=1}^{n} a_i = a_1+a_2+\cdots+a_n$

the $\underbrace{\text{remainder } R_n}_{\text{error using } S_n \text{ to approximate } S} = S - S_n$

$$\int_{n+1}^{\infty} f(x)\,dx \le \underbrace{R_n \le \int_{n}^{\infty} f(x)\,dx}$$

more important part since we usually want the maximum error & not the minimum error

$$S_n + \int_{n+1}^{\infty} f(x)\,dx \le S \le S_n + \int_{n}^{\infty} f(x)\,dx$$

$\sum_{n=1}^{\infty} a_n$

$\boxed{\text{Ex 1}}$

a) Show $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges

b) Find how many terms are needed to ensure the remainder is less than 10^{-3}

a) $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ p-series $p = 3/2 > 1$ converges

* we could do Integral test here, but that is harder & they didn't specify

b)
$$R_n \le \int_n^\infty f(x)\,dx$$
$$= \int_n^\infty \frac{1}{x^{3/2}}\,dx$$
$$= \lim_{t\to\infty} \int_n^t x^{-3/2}\,dx$$
$$= \lim_{t\to\infty} -2x^{-1/2}\Big|_n^t$$
$$= \lim_{t\to\infty} -2\left[\frac{1}{\sqrt{t}} - \frac{1}{\sqrt{n}}\right] = -2\left[0 - \frac{1}{\sqrt{n}}\right]$$
$$= \underbrace{\frac{2}{\sqrt{n}} < 10^{-3}}_{\text{we want } R_n < 10^{-3}}$$

$$\frac{2}{\sqrt{n}} < 10^{-3}$$

$$\frac{2}{\sqrt{n}} < \frac{1}{10^3}$$

$$2\cdot 10^3 < \sqrt{n}$$
$$2000 < \sqrt{n}$$
$$2000^2 < n$$

$$n = 4000000 + 1 = \boxed{4{,}000{,}001 \text{ terms}}$$

Pleasantly surprised that I didn't need a calculator for this one

Ex 2

a) Show $\sum_{n=1}^{\infty} \frac{1}{n^6}$ converges

b) How many terms are needed to ensure the remainder is less than 10^{-3}?

c) Using your work from b) find the lower & upper bounds of the exact value of this series

a) $\sum_{n=1}^{\infty} \frac{1}{n^6}$ p-series

$p = 6 > 1$ converges

b)
$$R_n \le \int_n^{\infty} f(x)\,dx$$
$$= \int_n^{\infty} \frac{1}{x^6}\,dx$$
$$= \lim_{t\to\infty} \int_n^t x^{-6}\,dx$$
$$= \lim_{t\to\infty} -\frac{1}{5}x^{-5}\Big|_n^t = \lim_{t\to\infty} -\frac{1}{5}\left[\frac{1}{t^5} - \frac{1}{n^5}\right]$$
$$= -\frac{1}{5}\left[0 - \frac{1}{n^5}\right] = \frac{1}{5n^5} < 10^{-3} = \frac{1}{10^3}$$
$$10^3 < 5n^5$$
$$1000 < 5n^5$$
$$200 < n^5$$

$200^{1/5} < n$

$2.88 < n$

$n = 3$, $\boxed{3 \text{ terms}}$

★ On a test, this problem would need to be adjusted so you could do it without a calculator

c) $S_n + \int_{n+1}^{\infty} f(x)\,dx \le S \le S_n + \int_{n}^{\infty} f(x)\,dx$

$S_3 + \int_{4}^{\infty} \frac{1}{x^6}\,dx \le S \le S_3 + \int_{3}^{\infty} \frac{1}{x^6}\,dx$

$1 + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{5}\cdot\frac{1}{4^5} \le S \le 1 + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{5}\cdot\frac{1}{3^5}$

Recall we just found $\int_{n}^{\infty} \frac{1}{x^6}\,dx = \frac{1}{5}\cdot\frac{1}{n^5}$

$$\boxed{1.017192 \le S \le 1.01782}$$

For fun: what's the maximum error using the first 15 terms to estimate the series?

$R_{15} \le \int_{15}^{\infty} f(x)\,dx = \boxed{\frac{1}{5}\cdot\frac{1}{15^3}}$

✱ See one of our 4.3 HW hints on our moodle page for an example of how to do this if our sum doesn't start at $n=1$. FYI: I think you should be able to do Integral Test on problems where n doesn't start at 1, but I wouldn't ask you to estimate those on a test.

Recall: If we have a convergent Alternating Series $R_n = |S - S_n| < C_{n+1}$

$\boxed{\text{Ex 3}}$

a) Show $\sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1}$ converges

b) How many terms must be summed so that the remainder is less than 10^{-4}?

a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1}$ $\quad C_n = \frac{1}{2n+1} \geq C_{n+1} = \frac{1}{2(n+1)+1}$

decreasing ✓

$\lim_{n\to\infty} C_n = \lim_{n\to\infty} \frac{1}{2n+1} = 0$ ✓

Converges by AST

b) $R_n = |S - S_n| < C_{n+1}$

$\sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1} = -\frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \dots$

* Just realized this won't be efficient for this problem

$C_{n+1} = \frac{1}{2(n+1)+1} = \frac{1}{2n+3} < \frac{1}{10^4}$ so $10^4 < 2n+3$

$$10000 < 2n+3$$

$$\frac{10000-3}{2} < n$$

$$4998.5 < n$$

$$\boxed{n = 4999 \text{ terms}}$$

Ex 4

a) Show $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4}$ converges

b) How many terms must be summed so that the remainder is less than 10^{-4}?

c) What is that estimate?

a) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4}$ $\quad C_n = \frac{1}{n^4} \geq C_{n+1} = \frac{1}{(n+1)^4}$

decreasing ✓

$$\lim_{n\to\infty} C_n = \lim_{n\to\infty} \frac{1}{n^4} = 0 \checkmark$$

Converges by AST

b) $R_n < C_{n+1}$

This one can be done 2 ways:

1. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4} = \frac{1}{1^4} - \frac{1}{2^4} + \frac{1}{3^4} - \frac{1}{4^4} + \frac{1}{5^4} - \frac{1}{6^4} + \frac{1}{7^4} - \frac{1}{8^4} + \frac{1}{9^4} - \frac{1}{10^4} + \frac{1}{11^4} + \cdots$

$C_{n+1} = \frac{1}{11^4}$ $\quad n+1 = 11$ $\quad \boxed{n = 10}$ 10 terms

1st term where its absolute value is less than 10^{-4}

2. $R_n < C_{n+1}$

so we can just find $C_{n+1} = \frac{1}{(n+1)^4}$

$$C_{n+1} < \frac{1}{10^4}$$

$$\frac{1}{(n+1)^4} < \frac{1}{10^4}$$

$$10^4 < (n+1)^4$$

If $n=9$, they'd be equal so $\boxed{n=10}$

this way seems more efficient

c) $\boxed{1 - \frac{1}{2^4} + \frac{1}{3^4} - \frac{1}{4^4} + \frac{1}{5^4} - \frac{1}{6^4} + \frac{1}{7^4} - \frac{1}{8^4} + \frac{1}{9^4} - \frac{1}{10^4}}$

Ex 5 Estimate the value of the following convergent series with an absolute error less than 10^{-3}

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4+1} \qquad C_n = \frac{1}{n^4+1} \qquad C_{n+1} = \frac{1}{(n+1)^4+1}$$

$$R_n < C_{n+1} = \frac{1}{(n+1)^4+1} < \frac{1}{10^3}$$

$$10^3 < (n+1)^4 + 1$$

$$1000 - 1 < (n+1)^4$$

$$999^{1/4} < n+1 \rightarrow 4.62 < n$$

$\boxed{n=5}$

$\boxed{-\frac{1}{2} + \frac{1}{2^4+1} - \frac{1}{3^4+1} + \frac{1}{4^4+1} - \frac{1}{5^5+1}}$