# First Order Linear Equations

Recall: A $1^{st}$ order linear equation is a d.e. of the form $\underbrace{a_1(x)}\frac{dy}{dx} + \underbrace{a_0(x)}y = \underbrace{b(x)}$

functions of the independent variable

## Technique:

1. Write the d.e. in standard form:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

2. Calculate $\mu(x) = e^{\int P(x)\,dx}$

3. Multiply both sides of the d.e. in standard form by $\mu(x)$

$$\underbrace{\mu(x)\frac{dy}{dx} + \mu(x)P(x)y} = \mu(x)Q(x)$$

$$\frac{d}{dx}\left[\mu(x)y\right] = \mu(x)Q(x)$$

★ In class we saw this comes from the product rule.

4. Integrate & Solve for $y$

Examples from Introduction to Differential Equations with Boundary Value Problems by Campbell & Haberman

Solve the d.e. If there is no initial condition given find the general solution

**Ex 1** $(x^2+1)y' + 2xy = 1$

★ 1st put into standard form $\frac{dy}{dx} + P(x)y = Q(x)$

$$y' + \frac{2x}{x^2+1}y = \frac{1}{x^2+1}$$

★ Find $\mu(x) = e^{\int P(x)\,dx}$

$$\mu(x) = e^{\int \frac{2x}{x^2+1}dx} \qquad u = x^2+1, \quad du = 2x\,dx$$

$$= e^{\int \frac{du}{u}}$$

$$= e^{\ln|u|} = |u|$$

$$= |x^2+1|$$

$$= x^2+1$$

In class we saw that we don't need a +C on this step.

★ multiply both sides of the d.e. by $\mu(x)$. It needs to be the d.e. in standard form for this to work.

$$\underbrace{(x^2+1)y' + \cancel{(x^2+1)}\frac{(2x)y}{\cancel{x^2+1}}}_{\frac{d}{dx}\left[(x^2+1)y\right]} = \frac{1}{\cancel{x^2+1}}(x^2+1)$$

★ If you haven't messed up, this will always be $\frac{d}{dx}[\mu(x)y]$ by the product rule

For this problem we are basically back to where we started but with the left side simplified.

$$\frac{d}{dx}\left[(x^2+1)y\right] = 1$$

$$(x^2+1)\,y = \int 1\,dx = x + C$$

$$\boxed{y = \frac{x+C}{x^2+1}}$$ General solution

**Ex 2** $\quad xy' = -3y + \frac{\sin x}{x^2} \qquad y(1)=0$

☆ Standard form: $y' = -\frac{3}{x}y + \frac{\sin x}{x^3}$

$$y' + \frac{3}{x}y = \frac{\sin x}{x^3}$$

☆ $\mu(x) = e^{\int P(x)dx}$

$$\mu(x) = e^{\int \frac{3}{x}dx} = e^{3\ln x} = e^{\ln x^3} = x^3$$

☆ Just as we don't need a $+C$, we really don't need to worry about the absolute value we usually get integrating $\frac{1}{x}$. This is because we just want any $\mu$ that will work.

☆ multiply by $\mu$

$$\frac{d}{dx}\left[\underbrace{x^3}_{\mu} y\right] = x^3\,\frac{\sin x}{x^3} = \sin x$$

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* Integrate

$$x^3 y = \int \sin x \, dx = -\cos x + C$$

$$y = \frac{-\cos x + C}{x^3}$$

$$y(1) = 0 = \frac{-\cos 1 + C}{1^3} \qquad C = \cos 1$$

$$\boxed{y = \frac{-\cos x + \cos 1}{x^3}}$$

* $\cos 1$ is probably not a value you know.
Be sure to divide C by $\mu$!

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**Ex 3** $y' + 4y = x \quad y(0) = 0$

* Already in standard form. Just need $\mu$

$$\mu(x) = e^{\int 4\,dx} = e^{4x}$$

* multiply both sides by $\mu$

$$\frac{d}{dx}\left[e^{4x} y\right] = x e^{4x}$$

* Integrate

$$e^{4x} y = \int x e^{4x}\,dx$$

Integration by parts!
LIATE

$u = x \quad v = \frac{1}{4}e^{4x}$
$du = dx \quad dv = e^{4x}\,dx$

$$= uv - \int v\,du$$
$$= \frac{x}{4}e^{4x} - \int \frac{1}{4}e^{4x}\,dx$$
$$e^{4x} y = \frac{x}{4}e^{4x} - \frac{1}{16}e^{4x} + C$$

$$y = \frac{x}{4} - \frac{1}{16} + \frac{C}{e^{4x}}$$

$$y(0) = 0 = \frac{0}{4} - \frac{1}{16} + \frac{C}{e^0} = -\frac{1}{16} + C \qquad C = \frac{1}{16}$$

$$\boxed{y = \frac{x}{4} - \frac{1}{16} + \frac{1/16}{e^{4x}} = \frac{x}{4} - \frac{1}{16} + \frac{1}{16}e^{-4x}}$$

* Both of these are fine

$\boxed{\text{Ex } 4}$ $\frac{dx}{dt} = 3(x-1)$, $x(0) = 8$

☆ This problem could be done either as a 1st order linear or as a separable d.e. We will do it both ways.

☆ As a 1st order linear, step 1 is to get it into standard form

$$\frac{dx}{dt} = 3x - 3$$

$$\frac{dx}{dt} - 3x = -3$$

☆ Find $\mu$

$$\mu(t) = e^{\int -3\,dt}$$
$$= e^{-3t}$$

☆ Be <u>very</u> careful finding your $\mu$ we always need to include any negatives or constants with our $P(t)$ where $\frac{dx}{dt} + P(t)x = Q(t)$

☆ Multiply by $\mu$

$$\frac{d}{dt}\left[e^{-3t}x\right] = -3e^{-3t}$$

☆ Integrate $e^{-3t}x = \int -3e^{-3t}\,dt = e^{-3t} + C$

$$x = 1 + Ce^{3t}$$

$x(0) = 8 = 1 + Ce^0 = 1 + C \quad C = 7$

$$\boxed{x = 1 + 7e^{3t}}$$

**Ex 4 (Take 2) as a separable**

$$\frac{dx}{dt} = 3(x-1)$$

★ Recall: separable equations are of the form $\frac{dx}{dt} = f(t)g(x)$

$$\int \frac{dx}{x-1} = \int 3\,dt$$

$$\ln|x-1| = 3t + C$$

$$|x-1| = e^{3t+C}$$

$$x - 1 = ke^{3t}$$

$$x = 1 + ke^{3t}$$

$$x(0) = 8$$

$$= 1 + ke^0 = 1 + k \quad k = 7$$

$$\boxed{x = 1 + 7e^{3t}}$$

✓

★ On a test & doing your hw pay attention to the directions. Sometimes you can choose how you want to solve a d.e.

Not all separable eqns are linear & not all linear eqns are separable.