

11] Jacobians: Ignore if you  
are not in 242-040 or 242-050

In polar coordinates, we found  $dA = r dr d\theta$  by utilizing the area of a sector. In spherical coordinates, we found  $dV = \rho^2 \sin\phi d\rho d\phi d\theta$  by using geometry again.

Jacobians give us a way to find  $dA$  and  $dV$  if we change our coordinate systems & can't easily use geometry.

In 2D:

$$\iint_D f(x,y) dA = \iint_S f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du dv$$

where  $\left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u}$

Jacobian  
of the  
transformation

det  
determinant

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**Ex 1:** Use  $x = r\cos\theta$  &  $y = r\sin\theta$  to find the Jacobian for polar coordinates

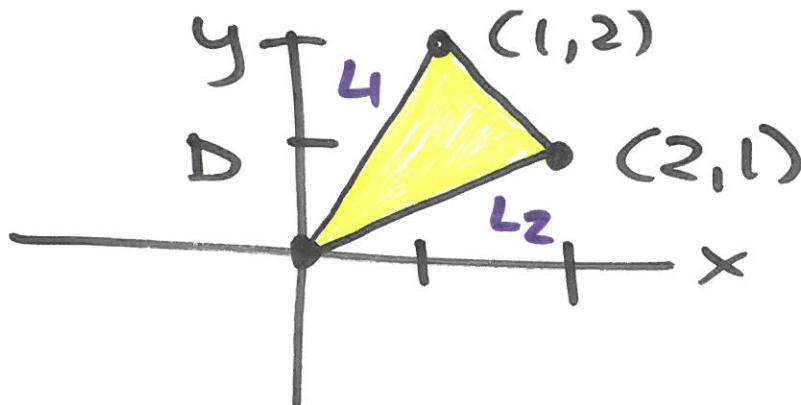
$$\left| \frac{\partial(x,y)}{\partial(r,\theta)} \right| = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta = r(1) = r$$

From Stewart Calculus:

**Ex 2:** Use the transformation  $x = 2u + v$ ,  $y = u + 2v$  to evaluate  $\iint_D (x - 3y) dA$  where  $D$  is the triangular region with vertices  $(0,0)$ ,  $(2,1)$ , &  $(1,2)$

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$$L_1: y = 2x, L_2: y = \frac{1}{2}x$$

Found these using  $y=mx+b$

$$y = 2x \rightarrow \underbrace{u+2v}_y = 2(\underbrace{2u+v}_x)$$

$$u+2v = 4u + 2v$$

$$0 = 3u$$

$$y = \frac{x}{2} \rightarrow u+2v = \frac{1}{2}(2u+v)$$

$$u+2v = u + \frac{v}{2}$$

$$2v = \frac{v}{2}$$

$$4v = v$$

$$v = 0$$

$L_3$ : the line including the points

(1, 2) & (2, 1)

$$m = \frac{\Delta y}{\Delta x} = \frac{1-2}{2-1} = -1$$

$$y = 3-x$$

$$y = -x + b$$

$$2 = -1 + b$$

$$b = 3$$

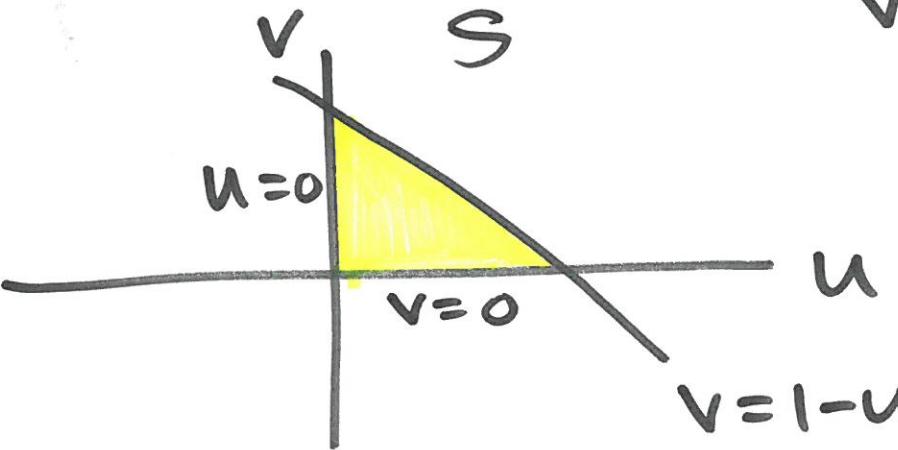
$$41 \quad y = 3 - x \rightarrow u + 2v = 3 - (2u + v)$$

$$u + 2v = 3 - 2u - v$$

$$3u + 3v = 3$$

$$u + v = 1$$

$$v = 1 - u$$



This is what  $D$  looks like on the  $uv$ -plane

$$\iint_D (x-3y) dA = \dots$$

Jacobian of our transformation =  $\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$

$$= \begin{vmatrix} \frac{\partial}{\partial u}(2u+v) & \frac{\partial}{\partial v}(2u+v) \\ \frac{\partial}{\partial u}(u+2v) & \frac{\partial}{\partial v}(u+2v) \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}$$

$$= 4 - 1 = 3$$

$\underline{5} \quad \iint_D (x-3y) dA =$   
 $\int_0^1 \int_0^{1-u} (2u+v - 3(u+2v)) |3| dx du$

$$\begin{aligned}
 & 3 \int_0^1 \int_0^{1-u} -u - 5v dv du \\
 &= 3 \int_0^1 -uv - \frac{5}{2} v^2 \Big|_0^{1-u} du \\
 &= 3 \int_0^1 -u(1-u) - \frac{5}{2} (1-u)^2 du \\
 &= 3 \int_0^1 -u + u^2 - \frac{5}{2} (1-2u+u^2) du \\
 &= 3 \int_0^1 -u + u^2 - \frac{5}{2} + 5u - \frac{5}{2} u^2 du \\
 &= 3 \left[ -\frac{1}{2}u^2 + \frac{1}{3}u^3 - \frac{5}{2}u + \frac{5}{2}u^2 - \frac{5}{6}u^6 \right]_0^1 \\
 &= 3 \left[ -\frac{1}{2} + \frac{1}{3} - \cancel{\frac{5}{2}} + \cancel{\frac{5}{2}} - \frac{5}{6} \right] \\
 &= 3 \left[ -\frac{6}{6} \right] = \boxed{-3}
 \end{aligned}$$

Note: original region  $D$  that would've needed to be split up to integrate.  
 problem had a

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**Ex 3**

Use the transformation

$$x = 2u, y = 3v \text{ to evaluate}$$

$$\iint_R x^2 dA \text{ where } R \text{ is bounded by the ellipse } 9x^2 + 4y^2 = 36$$

$$9x^2 + 4y^2 = 36 \rightarrow 9(2u)^2 + 4(3v)^2 = 36$$

$$36u^2 + 36v^2 = 36$$

$$u^2 + v^2 = 1$$



$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} = 6$$

$$\iint_R x^2 dA = \iint_S (2u)^2 |6| du dv$$

S is a circle. Switch to polar:

$$u = r \cos \theta, v = r \sin \theta$$

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$$\iint 4u^2 \cdot 6dA = \int_0^{2\pi} \int_0^1 24(r \cos \theta)^2 r dr d\theta$$

$$= \int_0^{2\pi} \cos^2 \theta d\theta \int_0^1 24r^3 dr$$

$$= \int_0^{2\pi} \left( \frac{1}{2}(1 + \cos 2\theta) \right) d\theta \quad 6r^4 \Big|_0^1$$

$$\frac{1}{2} \left[ \theta + \frac{1}{2} \sin 2\theta \right]_0^{2\pi} \quad 6$$

$$\frac{1}{2}[2\pi] 6 = \boxed{6\pi}$$

In 3D: If  $T$  is a transformation  
that maps a region  $S$  in  
 $uvw$ -space onto a region in  
 $xyz$ -space

$$x = g(u, v, w) \quad y = h(u, v, w)$$

$$z = k(u, v, w)$$

Jacobian of  $T$ :

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

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$$\iiint_E f(x, y, z) dV =$$

$$\iiint_S f(x(u, v, w), y(u, v, w), z(u, v, w)) \left| \frac{d(x, y, z)}{d(u, v, w)} \right| du dv dw$$

From Calculus by Briggs, Cochran,  
& Gillett:

**Ex 4** Use a change of variables to evaluate

$$\iiint_E xy dV \quad E \text{ is bounded by}$$

the planes  $y-x=0, y-x=2,$   
 $z-y=0, z-y=1, z=0, z=3$

\* I can see  $y-x$  &  $z-y$  appearing  
a few times.

$$u=y-x, v=z-y, w=z$$

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$$y-x=0 \rightarrow u=0$$

$$y-x=2 \rightarrow u=2$$

$$z-y=0 \rightarrow v=0$$

$$z-y=1 \rightarrow v=1$$

$$z=0 \rightarrow w=0$$

$$z=3 \rightarrow w=3$$

A parallelepiped in E becomes a rectangular box in S.

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

We need to find  $x, y, z$  in terms of  $u, v, w$ .

$$u=y-x \quad v=z-y \quad w=z$$

$$z=w \quad y=z-v=w-v \quad x=y-u=w-v-u$$

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$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} -1 & -1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= (-1) \begin{vmatrix} -1 & 1 \\ 0 & 1 \end{vmatrix} - (-1) \begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix} + 1 \begin{vmatrix} 0 & -1 \\ 0 & 0 \end{vmatrix}$$

\* We are doing the determinant of a  $3 \times 3$  matrix. This is the same thing we do with cross product

$$= -1(-1 \cdot 1 - 0 \cdot 1) - (-1)(0 \cdot 1 - 0 \cdot 1) + 1(0 \cdot 0 - 0)$$

$$= 1$$

Not sure why I find this deeply disappointing

Hard to read but still ultimately zero

$\int \int \int_E xy dV$   
 $= \int_0^3 \int_0^1 \int_0^2 (w-v-u)(w-v) \Big| \Big| \Big| du dv dw$   
 $\uparrow$   
 If this were negative, we'd find its absolute value.  
 $= \int_0^3 \int_0^1 \int_0^2 w^2 - \underbrace{wv - vw}_{-2wv} + v^2 - uw + uv du dv dw$   
 $= \int_0^3 \int_0^1 w^2 u - 2uwv + uv^2 - \frac{1}{2}u^2 w + \frac{1}{2}u^2 v \Big| \Big| \Big| dv dw$   
 $= \int_0^3 \int_0^1 2w^3 - 4wv + 2v^2 - 2w + 2v dw dv$   
 $= \int_0^3 2w^2 v - 2wv^2 + \frac{2}{3}v^3 - 2wv + v^2 \Big| \Big| \Big| dw$   
 $= \int_0^3 2w^2 - 2w + \frac{2}{3} - 2w + 1 dw$   
 $= \frac{2}{3}w^3 - w^2 + \frac{2}{3}w - w^2 + w \Big| \Big| \Big|_0^3 =$   
 $18 - 9 + 2 - 9 + 3 = \boxed{5}$