Calculus II Test 3 Version 1

1. (20 points) Solve the Initial Value Problem $y'' + 2y' + 10y = 90e^{5x}$; $y(0) = 7$, $y'(0) = 5$

2. (14 points) Use the differential equation $y'' + y' - 6y = f(x)$ along with the value of f(x) listed below to answer the following.

a) Find the complementary solution y_c

b) Find the form of y_p, but do **NOT** solve for the coefficients.

i) $f(x) = 11\sin(6x)$

ii) $f(x) = 7e^{-3x} + 2x$

3. (15 points) Initially 30 chipmunks live on an island. After one year, there are 50 chipmunks. Assume the chipmunk population on this island grows logistically and that the island can support up to 300 chipmunks. Find P(t), an equation for the population of chipmunks on the island after t years.

4. (13 points) Solve the IVP $y'' - 2y' + y = 0$, $y(0) = 3$, $y'(0) = 0$

5. (10 points) Use Euler's method with a stepsize of 0.5 to estimate y(1.5) and y(2) where y(x) is a solution to the IVP $\frac{dy}{dx} = 2x + y$; $y(1) = 2$

Clearly label your approximations. Note that y(2) doesn't need to be simplified.

6. (12 points) Find the orthogonal trajectories of $y = kx$. You can leave your answer as an implicit solution.

7. (16 points) A tank initially contains 80L of pure water. A brine solution containing 0.5 g of salt per liter enters the tank at a rate of 8L/min. The solution is well-mixed and drains from the tank at the same rate. Find y(t) the amount of salt in the tank at time t. What will happen as $t \to \infty$?

C2 T3 V1 Solutions

1. (20 pts) $r^2+2r+10=0$

$$r=\frac{-2\pm\sqrt{4-40}}{2}=\frac{-2\pm 6i}{2}=-1\pm 3i$$

$$y_c=e^{-x}[c_1\cos 3x+c_2\sin 3x]$$

$$y_p=Ae^{5x}$$

$$y_p'=5Ae^{5x}$$

$$y_p''=25Ae^{5x}$$

$$25Ae^{5x}+10Ae^{5x}+10Ae^{5x}=90e^{5x}$$

$$45A=90 \qquad A=2$$

$$y_p=2e^{5x}$$

$$y=e^{-x}[c_1\cos 3x+c_2\sin 3x]+2e^{5x}$$

$$y(0)=7=c_1+2 \qquad c_1=5$$

$$y=e^{-x}[5\cos 3x+c_2\sin 3x]+2e^{5x}$$

$$y'=-e^{-x}[5\cos 3x+c_2\sin 3x]+e^{-x}[-15\sin 3x+3c_2\cos 3x]+10e^{5x}$$

$$y'(0)=5=-1[5]+3c_2+10 \qquad c_2=0$$

$$\boxed{y=e^{-x}[5\cos 3x]+2e^{5x}}$$

2 (14pts)

$r^2 + r - 6 = 0$

$(r+3)(r-2) = 0$

a) $y_c = C_1 e^{-3x} + C_2 e^{2x}$

b) i) $y_p = A\cos 6x + B\sin 6x$

ii) $y_p = Ae^{-3x}x + Bx + C$

3. (15pts)

$$P(t) = \frac{300}{1 + Ae^{-rt}}$$

$$P(0) = 30 = \frac{300}{1+A}$$

$1 + A = 10 \quad A = 9$

$$P = \frac{300}{1 + 9e^{-rt}}$$

$$P(1) = 50 = \frac{300}{1 + 9e^{-r}}$$

$1 + 9e^{-r} = 6$

$-r = \ln(5/9)$

$$\boxed{P = \frac{300}{1 + 9e^{\ln(5/9)t}}}$$

4. (13 pts)

$r^2 - 2r + 1 = 0 \quad (r-1)^2 = 0$

$y = c_1 e^x + c_2 x e^x$

$y(0) = 3 = c_1$

$y' = 3e^x + c_2 e^x + c_2 x e^x$

$y'(0) = 0 = 3 + c_2 \quad c_2 = -3$

$$\boxed{y = 3e^x - 3xe^x}$$

5. (10 pts)

$y(1.5) \approx y_1 = y_0 + \Delta x f(x_0, y_0)$

$= 2 + 1/2 [2 \cdot 1 + 2]$

$= 4$

$y(2) \approx y_2 = y_1 + \Delta x f(x_1, y_1)$

$= 4 + 1/2 [2(1.5) + 4]$

6. (12 pts) $y = kx$

$\frac{dy}{dx} = k$

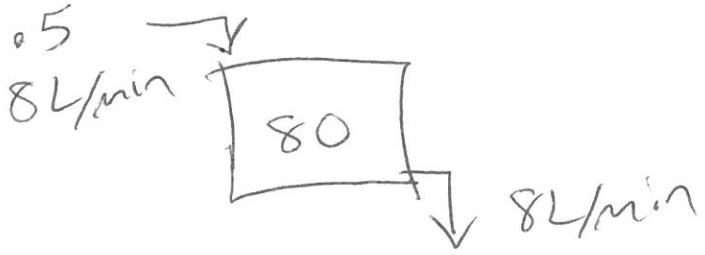
$\perp: \frac{dy}{dx} = -\frac{1}{k}$

$k: \; k = y/x \rightarrow \frac{dy}{dx} = -\frac{1}{\frac{y}{x}} = -\frac{x}{y}$

$\int y\,dy = \int -x\,dx$

$$\boxed{\frac{1}{2}y^2 = -\frac{1}{2}x^2 + C}$$

7. (16 pts)



$$\frac{dy}{dt} = F_i C_i - F_o C_o$$
$$= 8(.5) - 8\left(\frac{y}{80}\right)$$
$$= 4 - \frac{y}{10}$$
$$= -\frac{1}{10}(y-40)$$

$$\int \frac{dy}{y-40} = \int -\frac{1}{10}\,dt$$

$$\ln|y-40| = -\frac{1}{10}t + C$$

$$y-40 = ke^{-t/10}$$

$$y = 40 + ke^{-t/10}$$

$$y(0) = 0 = 40 + k$$

$$\boxed{y = 40 - 40e^{-t/10}}$$

$$\boxed{y \to 40 \text{ as } t \to \infty}$$