

# The Chain Rule

There are two main cases of the Chain Rule which we discussed in class.

**Case 1:**  $z = f(x, y)$  is a differentiable expression of  $x$  and  $y$  where  $x = x(t)$  and  $y = y(t)$  are both differentiable functions of  $t$  then

$$\frac{dz}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$$

\* Note:  $\frac{\partial f}{\partial x}$  and  $\frac{\partial f}{\partial y}$  are our only partial derivatives here.  $\frac{dz}{dt}$  is a regular derivative since if we wrote

$z = f(x(t), y(t))$  we would have a function of only  $t$ .

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If we have a curve  $\vec{r}(t) = \langle x(t), y(t) \rangle$   
then case 1 can also be  
represented as

$$\frac{d(f \circ \vec{r})}{dt} = \nabla f(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt}$$

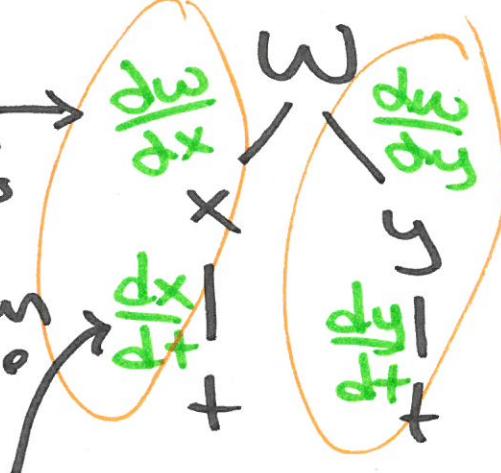
Recall: This is how we  
write the composition of two  
functions  $f \circ \vec{r} = f(\vec{r}(t))$

Examples from Calculus by  
Edwards & Penney  
or inspired by

**Ex 1:** Find  $\frac{dw}{dt}$  if  $w = e^{-x^2-y^2}$   
 $x=t, y=\sqrt{t}$

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partial derivatives  
since  $w$  is a function of more than one variable



what we're taking the derivative of  
what it is a function of  
what those are functions of

regular derivatives  
since  $x$  and  $y$  are only functions of one variable

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt}$$

$$\frac{dw}{dt} = \left( -2x e^{-x^2-y^2} \right) (1) + \left( -2y e^{-x^2-y^2} \right) \left( \frac{1}{2} t^{-\frac{1}{2}} \right)$$

$$\boxed{w = e^{-x^2-y^2}}$$

$$x = t$$

$$y = \sqrt{t}$$

Unless they say otherwise assume they want our answer only in terms of  $t$

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$$\begin{aligned}\frac{dw}{dt} &= \left(-2xe^{-x^2-y^2}\right) - 2ye^{-x^2-y^2}\left(\frac{1}{2}t^{-\frac{1}{2}}\right) \\ &= e^{-x^2-y^2} \left[ -2x - 2y\left(\frac{1}{2}t^{-\frac{1}{2}}\right) \right]\end{aligned}$$

$$\frac{dw}{dt} = e^{-t^2-t} \left[ -2t - \sqrt{t} t^{-\frac{1}{2}} \right]$$

$\uparrow$                                      $\uparrow$   
 $\frac{1}{\sqrt{t}}$

For this problem  $x=t, y=\sqrt{t}$

$$\boxed{\frac{dw}{dt} = e^{-t^2-t} \left[ -2t - 1 \right]}$$

**Ex 1.5:** Find  $\frac{d}{dt}(f \circ \vec{r})$  for

$$f(x,y) = e^{-x^2-y^2} \text{ where } \vec{r}(t) = \langle t, \sqrt{t} \rangle$$

Note: This is really the same problem just asked differently

$$\frac{d}{dt}(f \circ \vec{r}) = \nabla f(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt}$$

**51**  $\nabla f = \langle f_x, f_y \rangle = \langle -2x e^{-x^2-y^2}, -2y e^{-x^2-y^2} \rangle$

$$\nabla f(\vec{r}(t)) = \langle -2te^{-t^2-t}, -2\sqrt{t}e^{-t^2-t} \rangle$$

$$\vec{r}(t) = \langle t, \sqrt{t} \rangle = \langle t, t^{1/2} \rangle$$

$\begin{matrix} \uparrow & \uparrow \\ x=t & y=\sqrt{t} \end{matrix}$

$$\frac{d\vec{r}}{dt} = \langle 1, \frac{1}{2}t^{-1/2} \rangle$$

$$\begin{aligned} \frac{d}{dt}(f \circ \vec{r}) &= \langle -2te^{-t^2-t}, -2\sqrt{t}e^{-t^2-t} \rangle \cdot \langle 1, \frac{1}{2\sqrt{t}} \rangle \\ &= -2te^{-t^2-t} - 2\sqrt{t}e^{-t^2-t} \left( \frac{1}{2\sqrt{t}} \right) \end{aligned}$$

which is the same answer as before.

**Ex 2:** Find  $\frac{d}{dt}(f \circ \vec{r})$  if

$$f(x, y, z) = \sin(x^2yz) \quad \vec{r}(t) = \langle t, t^2, t^3 \rangle$$

$\begin{matrix} \uparrow & \uparrow & \uparrow \\ x & y & z \end{matrix}$

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$$f = \sin(x^2yz)$$

$$\nabla f = \langle f_x, f_y, f_z \rangle$$

$$= \langle 2x \cos(x^2yz)yz, x^2z \cos(x^2yz),$$

If there are common factors pulling them out can make it easier

$$= \cos(x^2yz) \langle 2xyz, x^2z, x^2y \rangle$$

$$\nabla f(\vec{r}(t)) = \cos(t^2 t^2 t^3) \langle 2t t^2 t^3, t^2 t^3, t^2 t^2 \rangle$$

$$= \cos(t^7) \langle 2t^6, t^5, t^4 \rangle$$

$$\vec{r}(t) = \langle t, t^2, t^3 \rangle$$

$$\frac{d}{dt} (f \circ \vec{r}) = \nabla f(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt}$$

$$=$$

$$\cos(t^7) \langle 2t^6, t^5, t^4 \rangle \cdot \langle 1, 2t, 3t^2 \rangle$$

$$= \cos(t^7) [2t^6 + 2t^6 + 3t^6]$$

$$= \boxed{\cos(t^7)(7t^6)}$$

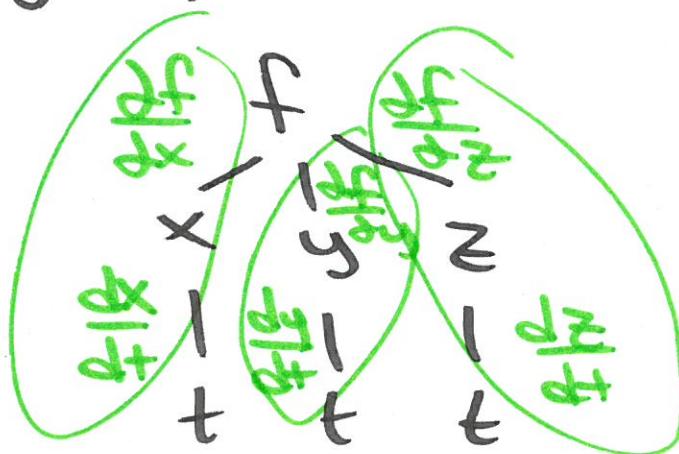
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**Ex 2.5:** Same problem, asked

differently: Find  $\frac{df}{dt}$  if

$$f(x, y, z) = \sin(x^2yz) \text{ where}$$

$$x=t, y=t^2, z=t^3$$



Tree Diagram

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} + \frac{\partial f}{\partial z} \frac{dz}{dt}$$



partials

$$\begin{aligned}
 \frac{df}{dt} &= 2xyz \cos(x^2yz)(1) + x^2z \cos(x^2yz)2t \\
 &\quad + x^2y \cos(x^2yz)3t^2 \\
 &= \cos(x^2yz) [2xyz + x^2z 2t + x^2y 3t^2] \\
 &= \cos(t^7) [2t^6 + 2t^6 + 3t^6]
 \end{aligned}$$

**Case 2:** Suppose that  $z = f(x, y)$  is a differentiable function of  $x$  and  $y$  where  $x = x(s, t)$ ,  $y = y(s, t)$  are differentiable functions of  $s \neq t$  then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s}$$

and

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t}$$

} all partials  
 Note:  
 $z = f(x(s, t), y(s, t))$   
 would still  
 be a function  
 of more than  
 one variable

Alternatively, if  $\vec{r}(s, t) = \langle x(s, t), y(s, t) \rangle$

$$\frac{d(f \circ \vec{r})}{ds} = \nabla f(\vec{r}(s, t)) \cdot \frac{d\vec{r}}{ds}$$

$$\frac{d(f \circ \vec{r})}{dt} = \nabla f(\vec{r}(s, t)) \cdot \frac{d\vec{r}}{dt}$$

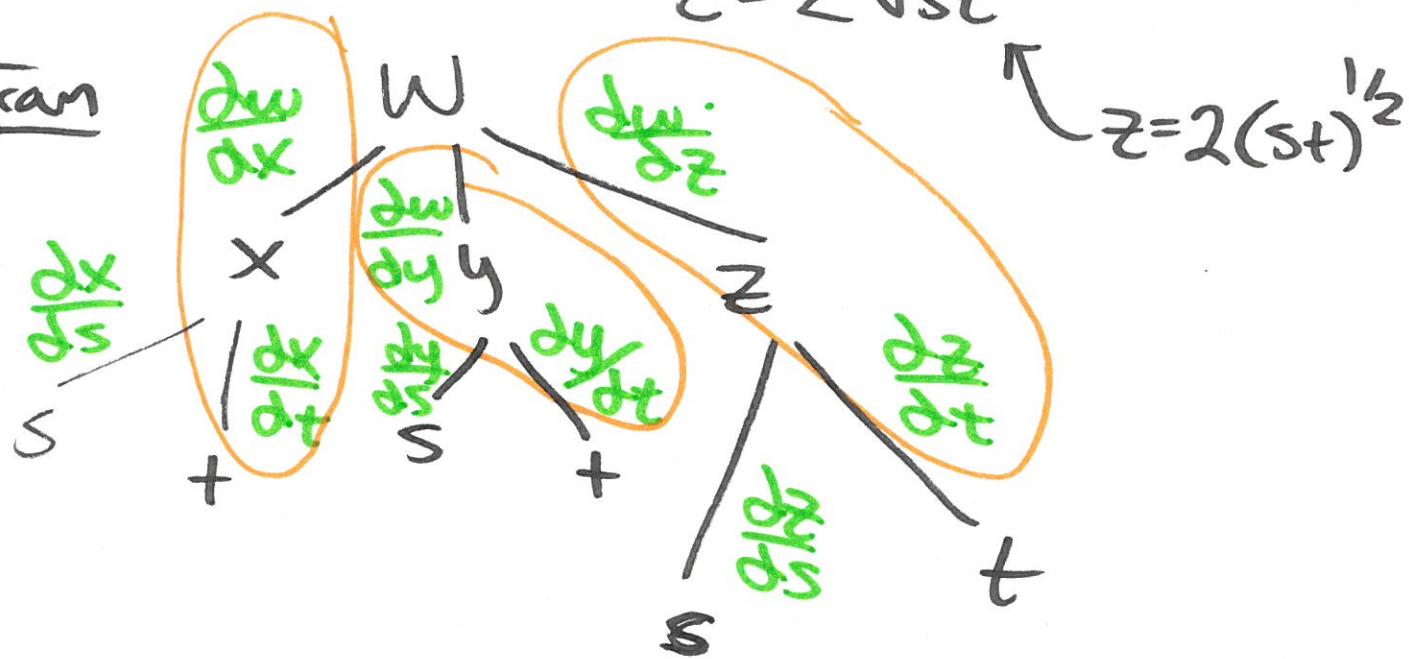
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**Ex 3:** Find  $\frac{dw}{dt}$  if

$$w = \ln(x^2 + y^2 + z^2) \quad x = s-t, y = s+t,$$

$$z = 2\sqrt{st}$$

Tree Diagram



$$\begin{aligned}
 \frac{dw}{dt} &= \frac{\partial w}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial t} \\
 &= \left( \frac{2x}{x^2 + y^2 + z^2} \right)(-1) + \left( \frac{2y}{x^2 + y^2 + z^2} \right)(1) + \left( \frac{2z}{x^2 + y^2 + z^2} \right) \left( \frac{s}{\sqrt{st}} \right) \\
 &= \frac{2}{x^2 + y^2 + z^2} \left( -x + y + 2z \frac{s}{\sqrt{st}} \right) \\
 &= \frac{2}{(s-t)^2 + (s+t)^2 + (2\sqrt{st})^2} \left[ -(s-t) + (s+t) + 2\sqrt{st} \frac{s}{2t} \right]
 \end{aligned}$$

Stop here

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$$\begin{aligned}
 \frac{\partial w}{\partial t} &= \frac{2}{s^2 - 2st + t^2 + s^2 + 2st + t^2 + 4st} [2t + 2s] \\
 &= \frac{2}{2s^2 + 4st + 2t^2} [2t + 2s] \\
 &= \frac{1}{s^2 + 2st + t^2} [2t + 2s] \\
 &= \frac{1}{(s+t)^2} [2(t+s)] \\
 &= \boxed{\frac{2}{s+t}}
 \end{aligned}$$

This was only because it looked like it simplified a lot & I was curious.

This could also be asked as

"find  $\frac{\partial}{\partial t}(w \circ \vec{r})$  of  $w = \ln(x^2, y^2, z^2)$

$$\vec{r}(s, t) = \langle s-t, s+t, 2\sqrt{st} \rangle$$

$$\frac{\partial}{\partial t}(w \circ \vec{r}) = \nabla w(\vec{r}) \cdot \frac{d\vec{r}}{dt}$$

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$$\boxed{\text{Ex 4}}: f(x,y,z) = \sqrt{x^2 + y^2 + z^2}$$

$$\vec{r}(u,v) = \langle 3e^u \sin v, 3e^u \cos v, 4e^u \rangle$$

$$\text{Find } \frac{\partial}{\partial u} (f \circ \vec{r})$$

\* As seen in both this example & the previous one, in a case 2 problem I'll only ask for one of the partial derivatives. For instance, here I'm not asking for  $\frac{\partial}{\partial v} (f \circ \vec{r})$  as well.

$$f = (x^2 + y^2 + z^2)^{1/2}$$

$$\nabla f = \langle f_x, f_y, f_z \rangle = \left\langle \frac{x}{\sqrt{x^2 + y^2 + z^2}}, \frac{y}{\sqrt{x^2 + y^2 + z^2}}, \frac{z}{\sqrt{x^2 + y^2 + z^2}} \right\rangle$$

$$\nabla f = \frac{1}{\sqrt{x^2 + y^2 + z^2}} \langle x, y, z \rangle$$

$$\nabla f(\vec{r}) = \frac{1}{\sqrt{(3e^u \sin v)^2 + (3e^u \cos v)^2 + (4e^u)^2}} \langle 3e^u \sin v, 3e^u \cos v, 4e^u \rangle$$

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$$\nabla f(\vec{r}) = \langle 3e^u \sin v, 3e^u \cos v, 4e^u \rangle$$

$$\frac{\sqrt{9e^{2u} \sin^2 v + 9e^{2u} \cos^2 v + 16e^{2u}}}{9e^{2u} (\sin^2 v + \cos^2 v)} \\ = \frac{1}{9e^{2u}}$$

$$= \langle 3e^u \sin v, 3e^u \cos v, 4e^u \rangle$$

$$\sqrt{25e^{2u}}$$

$$= \langle 3e^u \sin v, 3e^u \cos v, 4e^u \rangle$$

$$5e^u$$

$$= \left\langle \frac{3}{5} \sin v, \frac{3}{5} \cos v, \frac{4}{5} \right\rangle$$

$$\frac{d}{du} (f \circ \vec{r}) = \nabla f(\vec{r}) \cdot \frac{d\vec{r}}{du}$$

$$= \left\langle \frac{3}{5} \sin v, \frac{3}{5} \cos v, \frac{4}{5} \right\rangle \cdot \langle 3e^u \sin v, 3e^u \cos v, 4e^u \rangle$$

$\frac{d\vec{r}}{du} = \vec{r}$  for this problem

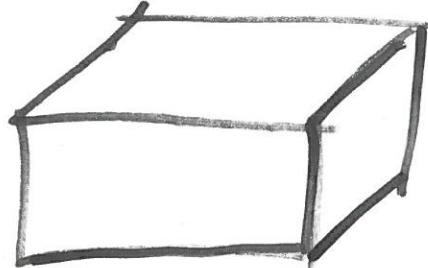
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$$\begin{aligned}\frac{\partial}{\partial u} (f \circ \vec{r}) &= \frac{9}{5} \sin^2 v e^u + \frac{9}{5} \cos^2 v e^u + \frac{16}{5} e^u \\&= \frac{9}{5} e^u (\sin^2 v + \cos^2 v) + \frac{16}{5} e^u \\&= \frac{9}{5} e^u (1) + \frac{16}{5} e^u \\&= \frac{25}{5} e^u = \boxed{5e^u}\end{aligned}$$

\* So long as you are correctly differentiating & plugging in, I don't need this level of simplifying. That said, the problems from this textbook are impressive.

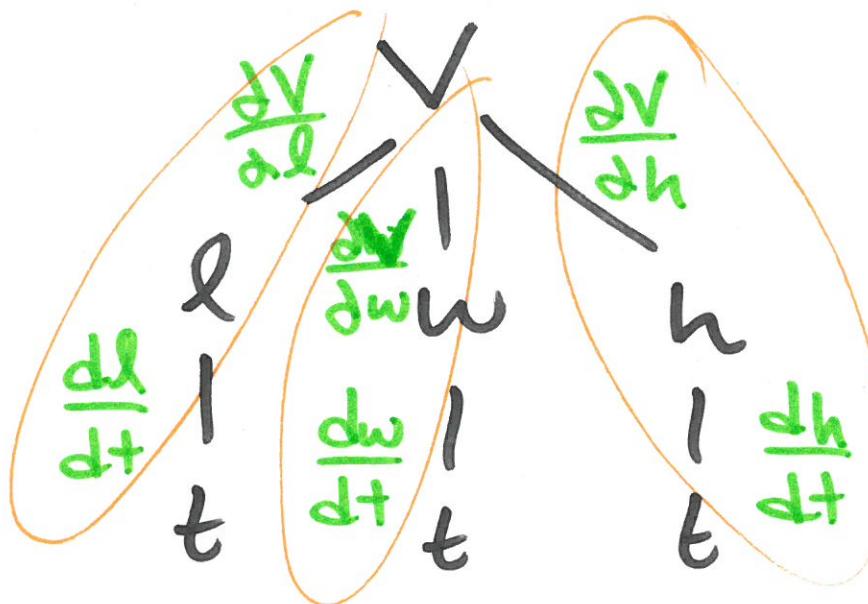
**Ex 5** The sun is melting a rectangular block of ice. When the block's height is 1 ft and the edge of its square base is 2 ft, its height is decreasing at 2 in/hr and its base edge is decreasing at 3 in/hr. What is the block's rate of change of volume V at that instant?

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$$V = lwh$$

Square base  $\rightarrow l=w$



**Case 1:**  
Since  $l, w, \& h$   
depend on  
time  $t$  not  
multiple  
variables

$$\frac{dV}{dt} = \frac{\partial V}{\partial l} \frac{dl}{dt} + \frac{\partial V}{\partial w} \frac{dw}{dt} + \frac{\partial V}{\partial h} \frac{dh}{dt}$$

$$= (wh) \frac{dl}{dt} + (lh) \frac{dw}{dt} + (lw) \frac{dh}{dt}$$

$$= (2 \cdot 1) \left(-\frac{1}{4}\right) + (2 \cdot 1) \left(-\frac{1}{4}\right) + (2 \cdot 2) \left(-\frac{1}{2}\right)$$

$$= -\frac{1}{2} - \frac{1}{2} - \frac{2}{3}$$

$$= \boxed{-\frac{5}{3} \text{ ft}^3/\text{hr}}$$

$$= -\frac{2880 \text{ in}^3}{\text{hr}}$$

$$-3 \sin \cdot \frac{1 \text{ ft}}{12 \text{ in}} = -\frac{3}{12} \text{ ft}$$

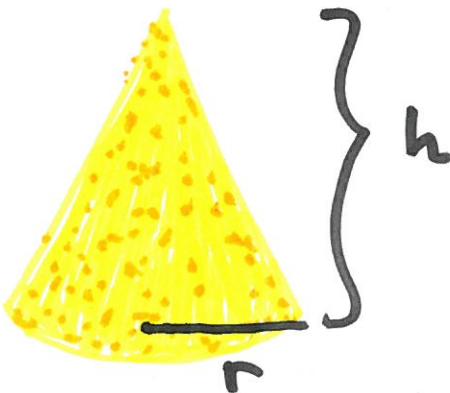
↑ decreasing

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**Ex 6:** Falling sand forms a

conical sandpile. When the sandpile has a height of 5 ft & its base radius is 2 ft, its height is increasing at 0.4 ft/min & its base radius is increasing at  $\frac{0.7 \text{ ft}}{\text{min}}$ . At what rate is the volume of the sandpile increasing at that moment?

$$\text{Hint: } V = \frac{1}{3}\pi r^2 h$$



$$\begin{aligned}
 \frac{dV}{dt} &= V_r \frac{dr}{dt} + V_h \frac{dh}{dt} \\
 &= \frac{2}{3}\pi rh \frac{dr}{dt} + \frac{1}{3}\pi r^2 \frac{dh}{dt} \\
 &= \frac{2}{3}\pi(2)(5)(0.7) + \frac{1}{3}\pi 2^2(0.4) \\
 &= \frac{14\pi}{3} + \frac{8\pi}{15} = \frac{70\pi}{15} + \frac{8\pi}{15} = \frac{78\pi}{15} = \boxed{\frac{26\pi}{5} \frac{\text{ft}^3}{\text{min}}}
 \end{aligned}$$