1) # The Phase Plane

Starting with an autonomous system of equations

$$\frac{dx}{dt} = f(x,y)$$
$$\frac{dy}{dt} = g(x,y)$$

(autonomous so no ts present on the right hand side)

The phase plane equation is $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

By solving the phase plane equation we can learn how x & y relate to each other

2) From 5.4 in your textbook:

Solve the phase plane equation

7. $\frac{dx}{dt} = y-1 \quad \frac{dy}{dt} = e^{x+y}$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{e^{x+y}}{y-1}$$

☆ Frequently when we are asked to solve the phase plane equation, we'll get a separable d.e. But as we'll see in the next example that isn't the only possibility.

$$\frac{dy}{dx} = \frac{e^x e^y}{y-1}$$

$$\int \frac{(y-1)}{e^y}\,dy = \int e^x\,dx$$

3) $\int (y-1)e^{-y}\,dy = \int e^{x}\,dx$

Integration by parts!

$\int u\,dv = uv - \int v\,du$

Choose u by LIATE

$u = y-1 \quad v = -e^{-y}$

$du = dy \quad dv = e^{-y}\,dy$

$uv - \int v\,du$

$(y-1)(-e^{-y}) - \int -e^{-y}\,dy$

$(y-1)(-e^{-y}) - e^{-y}$

$-e^{-y}(y \cancel{-1} \cancel{+1})$

$\underbrace{-ye^{-y}}_{\text{Not as bad as I'd thought it would be}} = \int e^{x}\,dx = e^{x} + C$

$\boxed{-ye^{-y} = e^{x} + C}$

implicit solution, can't solve for y.

4)

9. $\frac{dx}{dt} = 2y - x$

$\frac{dy}{dt} = e^x + y$

$\frac{dy}{dx} = \frac{e^x + y}{2y - x}$ not separable

$(2y-x)\,dy = (e^x + y)\,dx$

This looks kind of like an exact equation:

Recall in 2.4 $M(x,y)\,dx + N(x,y)\,dy = 0$

is exact if $M_y = N_x$

$\underbrace{-(e^x + y)}_{M}\,dx + \underbrace{(2y-x)}_{N}\,dy = 0$

$M_y = -1 = N_x$ It is exact!

5) $M = F_x$ $\qquad N = F_y$

$F_x = -e^x - y$ $\qquad F_y = 2y - x$

$\downarrow$ antiderivative wrt x

$F = -e^x - xy + g(y)$

$\downarrow$ derivative wrt y

$F_y = 0 - \underline{x} + g'(y) = 2y - \underline{x}$

$g'(y) = 2y$

$g(y) = y^2 + C_1$

$F = -e^x - xy + y^2$

Solutions to exact equations are of the form $F(x,y) = C$

$\boxed{-e^x - xy + y^2 = C}$ another implicit solution

6) Solve the phase plane equation & sketch by hand several trajectories with their flow arrows

13. $\frac{dx}{dt} = (y-x)(y-1)$

$\frac{dy}{dt} = (x-y)(x-1)$

$$\frac{dy}{dx} = \frac{(x-y)(x-1)}{(y-x)(y-1)} = \frac{\cancel{(x-y)}(x-1)}{-\cancel{(x-y)}(y-1)}$$

$$\int (y-1)\,dy = \int -(x-1)\,dx$$

$$\frac{1}{2}y^2 - y = -\frac{1}{2}x^2 + x + C$$

$$\frac{1}{2}y^2 - y + \frac{1}{2}x^2 - x = C$$

7) mult by 2

$$y^2 - 2y + x^2 - 2x = C_1$$

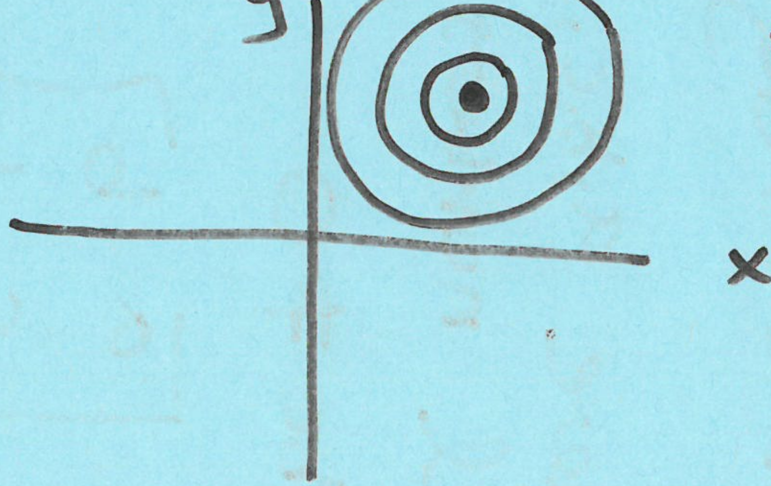
$$\boxed{(y-1)^2 + (x-1)^2 = C_2}$$

$$y^2 - 2y + \underbrace{1 + x^2 - 2x + 1}$$

adding 2 to the left when we complete the square so technically

$$C_2 = C_1 + 2$$

Circle centered at (1,1) with radius $\sqrt{C_2}$



Doesn't have flow arrows yet!

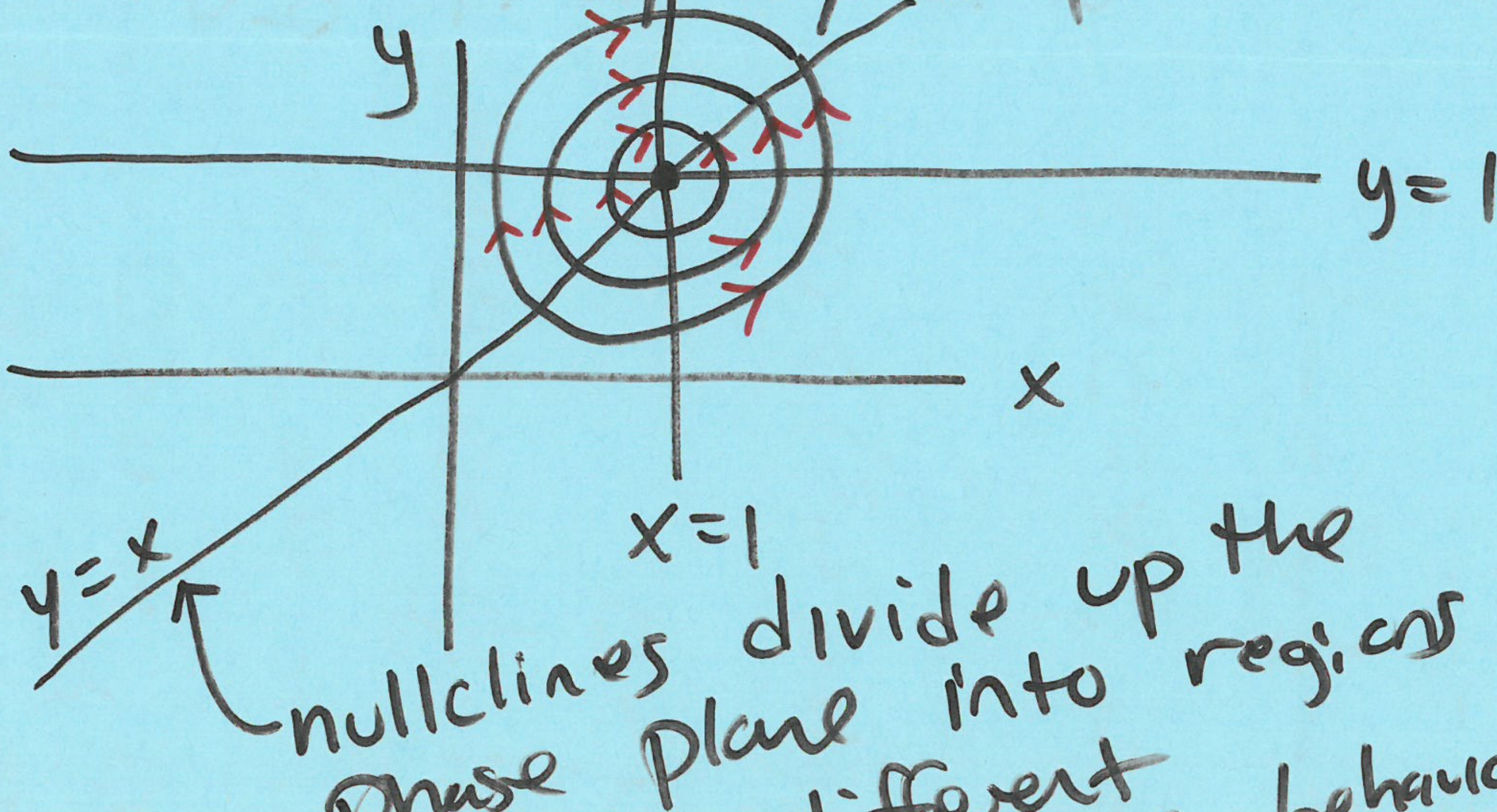
8) Note: the nullclines (where $\frac{dx}{dt}=0$ or $\frac{dy}{dt}=0$) are

$$\frac{dx}{dt} = (y-x)(y-1) = 0$$

$$y = x \ , \ y = 1$$

$$\frac{dy}{dt} = (x-y)(x-1)$$

$$y = x \ , \ x = 1$$



nullclines divide up the phase plane into regions of different qualitative behavior

9) If $\frac{dx}{dt} > 0$, arrows go to the right

$\frac{dx}{dt} < 0$, arrows go to the left

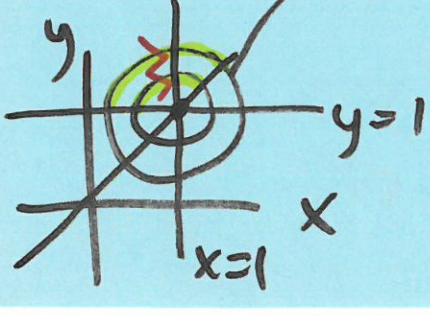
Likewise if $\frac{dy}{dt} > 0$, arrows go up. $\frac{dy}{dt} < 0$, arrows go down.

This problem is going to be a pain, unfortunately.

$$\frac{dx}{dt} = (y-x)(y-1) > 0$$

if $y-x > 0$ and $y-1 > 0$

or $y-x < 0$ and $y-1 < 0$

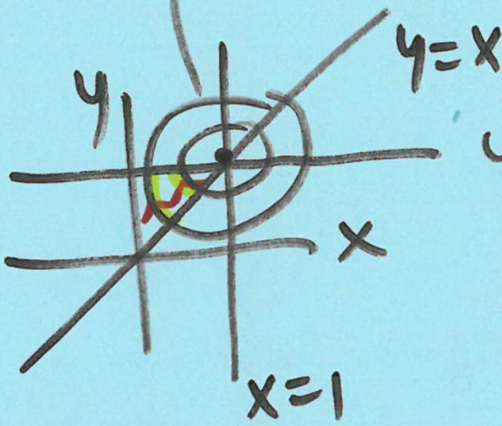


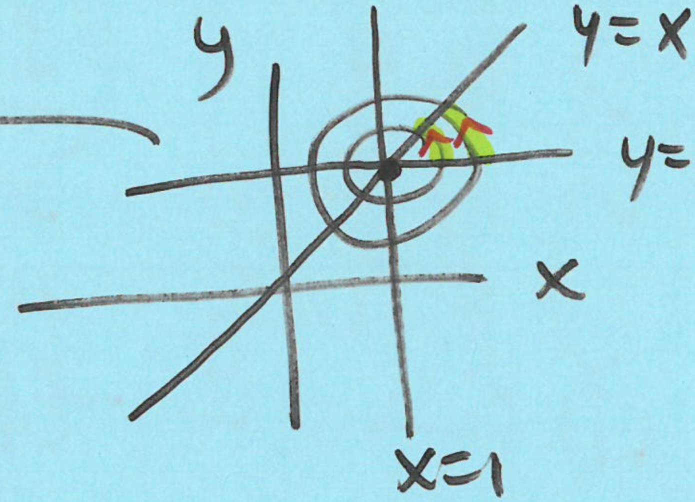
10)

$$\frac{dx}{dt} = (y-x)(y-1) < 0$$

If $y-x>0$ and $y-1<0$

or $y-x<0$ and $y-1>0$





* If this isn't clear you could also look at $\frac{dy}{dt}$

I won't ask a problem this complicated. Check out this related simpler problem

Ex $\frac{dx}{dt} = y-1 \quad \frac{dy}{dt} = -x+1$

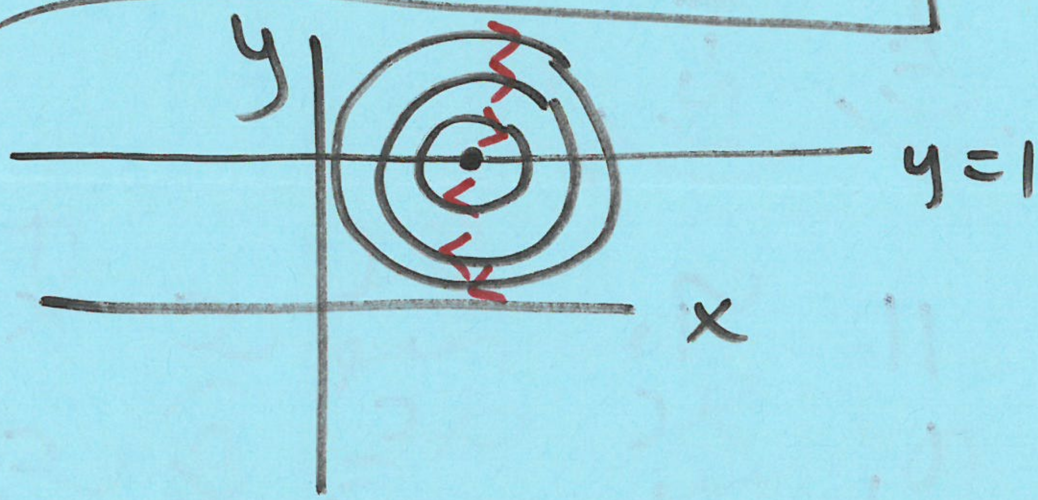
11) $\dfrac{dy}{dx} = \dfrac{-x+1}{y-1}$

$$\int y-1\,dy = \int -x+1\,dx$$

$$\frac{1}{2}y^2 - y = -\frac{1}{2}x^2 + x + C$$

$$\vdots$$

$$\boxed{(y-1)^2 + (x-1)^2 = C_2}$$



$$\frac{dx}{dt} = y-1 \;\begin{array}{ll} > 0 & \text{when } y>1 \\ < 0 & \text{when } y<1 \end{array}$$

(1,1) is a stable center

12) In 12.2 we can classify equilibrium points when we can't solve the phase plane equation if

$$\frac{dx}{dt} = ax + by$$

$$\frac{dy}{dt} = cx + dy$$

To do this we find the eigenvalues of A where

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ (so $|A - rI| = 0$)

Roots	Type	Stability
distinct, +	improper node	Unstable
distinct, −	improper node	asymptotically stabe
opposite signs	saddle point	unstable
equal, +	proper or improper node	unstable
equal, −	proper or improper node	asymptotically stable
$r = \alpha \pm \beta i$, $\alpha > 0$	Spiral pt	Unstable
$r = \alpha \pm \beta i$, $\alpha < 0$	spiral pt	asymptotically stable
$r = \pm \beta i$	center	stable

You won't be given this

13) From 12.2: Classify the critical point at the origin

1. $\frac{dx}{dt} = 5x + 6y$

$\frac{dy}{dt} = -5x - 8y$

☆ Find eigenvalues $A = \begin{bmatrix} 5 & 6 \\ -5 & -8 \end{bmatrix}$

$$|A - rI| = \begin{vmatrix} 5-r & 6 \\ -5 & -8-r \end{vmatrix}$$

$$= (5-r)(-8-r) + 30$$

$$= r^2 + 8r - 5r - 40 + 30 = 0$$

$$= r^2 + 3r - 10 = 0$$

$$= (r+5)(r-2) = 0$$

$r_1 = -5$, $r_2 = 2$. opposite signs

so $\boxed{(0,0) \text{ is an unstable saddle pt}}$

☆ The previous problem would have been hard to do by solving the phase plane eqn

14) Look at this simpler problem $\frac{dx}{dt} = -x$

$$\frac{dy}{dt} = 2y$$

Using the table: $\begin{vmatrix} -1-r & 0 \\ 0 & 2-r \end{vmatrix} =$

$$(-1-r)(2-r) = 0$$

$r_1 = -1$, $r_2 = 2$ opposite signs $\rightarrow$ unstable saddle

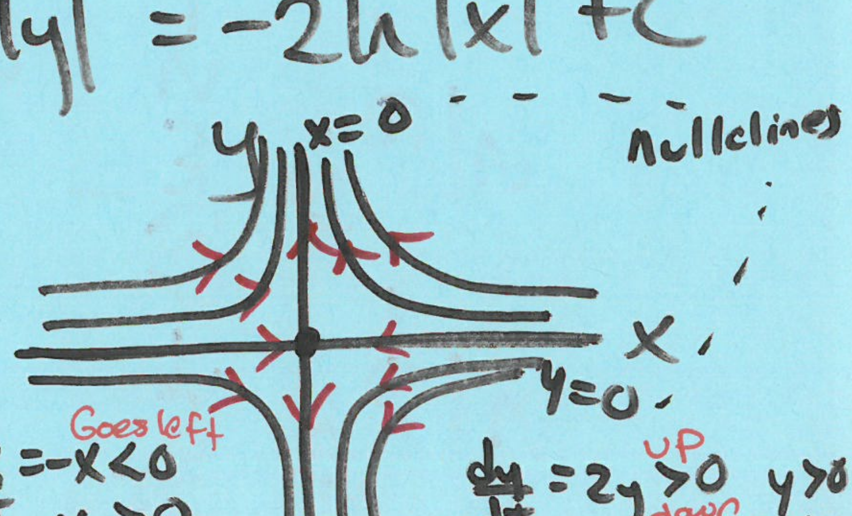
Using the phase plane eqn:

$\frac{dy}{dx} = \frac{2y}{-x}$ $\displaystyle\int \frac{dy}{y} = \int -\frac{2}{x}\,dx$

$$\ln|y| = -2\ln|x| + C$$

$$|y| = e^{-2\ln|x| + C}$$

$$\boxed{y = kx^{-2}}$$



Goes right $\frac{dx}{dt} = -x > 0$, $x < 0$

Goes left $\frac{dx}{dt} = -x < 0$, $x > 0$

$\frac{dy}{dt} = 2y$ up > 0 $y > 0$, down < 0 $y < 0$

15) We can also see it is an unstable saddle pt at (0,0) from our picture.

You should be able to do both.

5. $\frac{dx}{dt} = -6x + 3y$

$\frac{dy}{dt} = x - 4y$

$$A = \begin{bmatrix} -6 & 3 \\ 1 & -4 \end{bmatrix}$$

$$|A - rI| = \begin{vmatrix} -6-r & 3 \\ 1 & -4-r \end{vmatrix}$$

$$(-6-r)(-4-r) - 3 = 0$$

$$24 + 4r + 6r + r^2 - 3 =$$

$$r^2 + 10r + 21 = 0$$

$$(r+7)(r+3) = 0 \quad r_1 = -7, \; r_2 = -3$$

16) (0,0) is an asymptotically stable improper node.

☆ Look at this simpler problem

Ex $\frac{dx}{dt} = -6x$

$\frac{dy}{dt} = -4y$

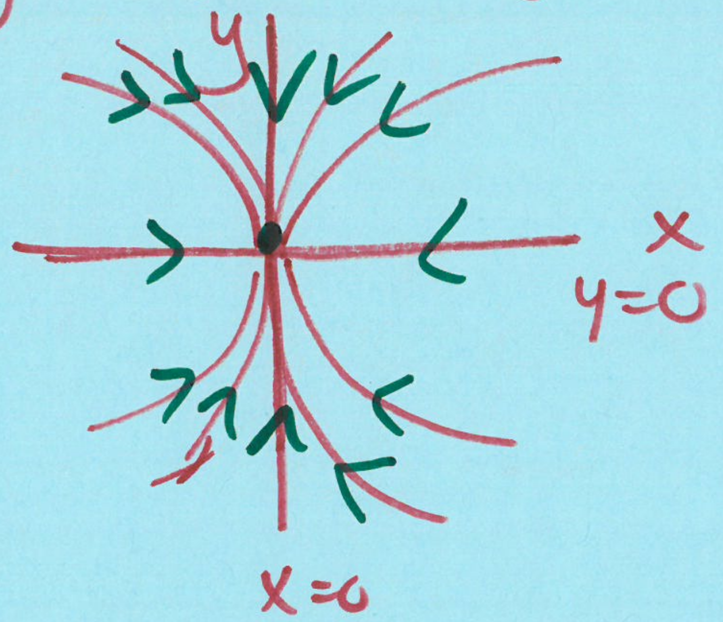
we can do it with the table using eigenvalues <u>but</u> we can also classify the critical point by solving the phase plane eqn.

$$\frac{dy}{dx} = \frac{-4y}{-6x} = \frac{2y}{3x}$$

$$\int \frac{1}{y} dy = \int \frac{2}{3} \frac{1}{x} dx$$

$$\ln|y| = \frac{2}{3} \ln|x| + C$$

17) $y = kx^{2/3}$ (I don't expect you to be able to graph this. You should be able to graph $y=x$, $y=x^2$, $y=\frac{1}{x}$, $y=\frac{1}{x^2}$, etc)



For different values of k this sort of looks like $x = \pm y^2$

$\frac{dx}{dt} = -6x > 0 \quad x < 0$

$\qquad\qquad < 0 \quad x > 0$

$\frac{dy}{dt} = -4y > 0 \quad y < 0$

asymptotically stable improper node.

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For critical points that are not at the origin, we need to shift them to the origin.

If (a,b) is a critical pt

use $x = u + a$ $\quad y = v + b$

Note! $\frac{dx}{dt} = \frac{du}{dt}$ $\quad \frac{dy}{dt} = \frac{dv}{dt}$

9. Find & classify the critical point of the given linear system

$$\frac{dx}{dt} = 2x + y + 9$$

$$\frac{dy}{dt} = -5x - 2y - 22$$

19a) We find the critical point by setting $\frac{dx}{dt}=0$ & $\frac{dy}{dt}=0$

$$2x+y+9=0 \xrightarrow{\text{mult by 2}} 4x+2y+18=0$$

$$-5x-2y-22=0 \qquad \begin{array}{r} -5x-2y-22=0 \\ \hline -x-4=0 \\ x=-4 \end{array}$$

Plug into one of the eqns

$$2(-4)+y+9=0$$

$$y=-1$$

critical pt $(-4,-1)$

$$x=u-4 \qquad y=v-1$$

Plug these into original d.e.

$$\frac{du}{dt}=2(u-4)+(v-1)+9$$

$$\frac{dv}{dt}=-5(u-4)-2(v-1)-22$$

20) Simplify the new d.e.

$$\frac{du}{dt} = 2u - 8 + v - 1 + 9$$
$$= 2u + v$$

$$\frac{dv}{dt} = -5u + 20 - 2v + 2 - 22$$
$$= -5u - 2v$$

★ your simplified eqns shouldn't have constants

$$A = \begin{bmatrix} 2 & 1 \\ -5 & -2 \end{bmatrix}$$

coefficients of u

coefficients of v

$$\begin{vmatrix} 2-r & 1 \\ -5 & -2-r \end{vmatrix} = (2-r)(-2-r) + 5 = 0$$
$$r^2 - 4 + 5 = 0$$
$$r^2 + 1 = 0$$

21) $r^2 = -1$

$r = \pm\sqrt{-1} = \pm i$

Look at chart, this is case 3 w/ $\alpha = 0$, $\boxed{\text{stable center at } (-4,-1)}$

☆ If asked,

We can actually solve the phase plane for this problem

$$\frac{dx}{dt} = 2x + y + 9$$

$$\frac{dy}{dt} = -5x - 2y - 22$$

$$\frac{dy}{dx} = \frac{-5x-2y-22}{2x+y+9}$$

$$(2x+y+9)\,dy = (-5x-2y-22)\,dx$$

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$$-(-5x-2y-22)dx+(2x+y+9)dy=0$$

$$\underbrace{(5x+2y+22)}_{M}dx+\underbrace{(2x+y+9)}_{N}dy=0$$

$$M_y=2=N_x \quad \text{exact!}$$

$F_x=M \qquad F_y=N$

$F_x=5x+2y+22 \qquad F_y=2x+y+9$

$$F=\frac{5}{2}x^2+2xy+22x+g(y)$$

$$F_y=0+\underline{2x}+0+g'(y)=\underline{2x}+y+9$$

$$g'(y)=y+9$$

$$g(y)=\frac{1}{2}y^2+9y$$

23) Solutions to exact eqns are of the form $F(x,y) = C$

$$\boxed{\frac{5}{2}x^2 + 2xy + 22x + \frac{1}{2}y^2 + 9y = C}$$

not super helpful in this form but neat we were able to do it.