Classification of PDEs

Given a 2nd order PDE of the form (☆) $Au_{xx} + Bu_{xt} + Cu_{tt} + F(x,t,u,u_x,u_t) = 0$
Classify it as a hyperbolic, parabolic, or elliptic d.e.

$$\boxed{\begin{array}{l} D = B^2 - 4AC \\ \\ D > 0, \text{ hyperbolic} \\ D = 0, \text{ parabolic} \\ D < 0, \text{ elliptic} \end{array}}$$

In the case of A, B, & C all constant the following transformations give us easier d.e.

2) $Au_{xx} + Bu_{xt} + Cu_{tt} + F(x, t, u, u_x, u_t) = 0$

$D = B^2 - 4AC > 0$ Hyperbolic

Characteristic coordinates

$\xi = x + \left(\frac{-B+\sqrt{D}}{2C}\right)t \qquad \tau = x + \left(\frac{-B-\sqrt{D}}{2C}\right)t$

transforms (*) into

$u_{\xi\tau} + G(\xi, \tau, u, u_\xi, u_\tau) = 0$

If $C=0$, either use the transformation

$\xi = \left(\frac{-B+\sqrt{D}}{2A}\right)x + t \qquad \tau = \left(\frac{-B-\sqrt{D}}{2A}\right)x + t$

or

$v = u_x$ & solve from there

Examples from or inspired by An Introduction to Partial Differential Equations by Pinchover & Rubinstein

Ex 1 Classify $u_{xx} + 6u_{xy} + 5u_{yy} = 0$ as hyperbolic, parabolic, or elliptic & solve

- They've only given us spatial variables so (A) becomes $Au_{xx} + Bu_{xy} + Cu_{yy} + F(\dots) = 0$

$A = 1$, $B = 6$, $C = 5$

$D = B^2 - 4AC = 36 - 20 = 16 > 0$ Hyperbolic!

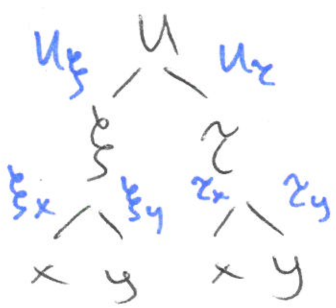
every t goes to y

$$\xi = x + \left(\frac{-B+\sqrt{D}}{2C}\right)y \qquad \tau = x + \left(\frac{-B-\sqrt{D}}{2C}\right)y$$

$$= x + \left(\frac{-6+4}{10}\right)y \qquad = x + \left(\frac{-6-4}{10}\right)y$$

$$= x - \frac{2}{10}y \qquad = x - y$$

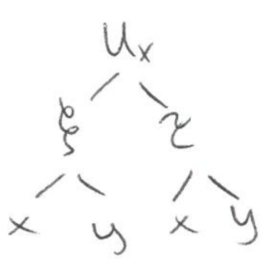
- Use Chain rule & plug back in



$$u_x = u_\xi \xi_x + u_\tau \tau_x = u_\xi + u_\tau$$

$\frac{\partial}{\partial x}\left(x - \frac{2}{10}y\right)$ $\quad \frac{\partial}{\partial x}(x-y)$

u_x is a function of ξ & τ so



$$u_{xx} = [u_x]_x = [u_x]_\xi \xi_x + [u_x]_\tau \tau_x$$

$$= [u_\xi + u_\tau]_\xi 1 + [u_\xi + u_\tau]_\tau 1$$

$$= u_{\xi\xi} + 2u_{\tau\xi} + u_{\tau\tau}$$

Likewise $u_{xy} = [u_x]_y = \underbrace{[u_x]_\xi}\, \xi_y + \underbrace{[u_x]_\eta}\, \eta_y$

we found these in the last step

$$= (u_{\xi\xi} + u_{\eta\xi})\left(-\tfrac{2}{10}\right) + (u_{\xi\eta} + u_{\eta\eta})(-1)$$
$$= -\tfrac{2}{10} u_{\xi\xi} - \tfrac{12}{10} u_{\eta\xi} - u_{\eta\eta}$$

$$u_y = u_\xi \xi_y + u_\eta \eta_y$$
$$= u_\xi\left(-\tfrac{2}{10}\right) + u_\eta(-1)$$

$$u_{yy} = [u_y]_y = [u_y]_\xi\left(-\tfrac{2}{10}\right) + [u_y]_\eta(-1)$$
$$= \left[u_\xi\left(-\tfrac{2}{10}\right) - u_\eta\right]_\xi\left(-\tfrac{2}{10}\right) + \left[u_\xi\left(-\tfrac{2}{10}\right) - u_\eta\right]_\eta(-1)$$
$$= \tfrac{4}{100} u_{\xi\xi} + \underbrace{\tfrac{2}{10} u_{\eta\xi} + \tfrac{2}{10} u_{\xi\eta}} + u_{\eta\eta}$$

$$u_{xx} + 6u_{xy} + 5u_{yy} = 0$$

$$(u_{\xi\xi} + 2u_{\eta\xi} + u_{\eta\eta}) + 6\left(-\tfrac{2}{10} u_{\xi\xi} - \tfrac{12}{10} u_{\eta\xi} - u_{\eta\eta}\right) + 5\left(\tfrac{4}{100} u_{\xi\xi} + \tfrac{4}{10} u_{\eta\xi} + u_{\eta\eta}\right) = 0$$

$u_{\xi\xi}$ coefficients $= 1 - \tfrac{12}{10} + \tfrac{20}{100} = 0$

$u_{\eta\eta}$ coeff $= 1 - 6 + 5 = 0$

$u_{\xi\eta} = u_{\eta\xi}$ coeff $\rightarrow 2 - \tfrac{72}{10} + \tfrac{20}{10} = \tfrac{-32}{10}$

so $u_{xx} + 6u_{xy} + 5u_{yy} = 0$ becomes

$$\tfrac{-32}{10} u_{\xi\eta} = 0$$
$$u_{\xi\eta} = 0$$

5 Note the canonical form for B^2-4AC, where $C \neq 0$ is

$$u_{\xi\eta} + G(\xi, \eta, u, u_\xi, u_\eta) = 0$$

our F was 0 $\left(A_{xx} + B_{xy} + C_{yy} + F(\dots) = 0\right)$

so our equation can just simplify to

$$\boxed{u_{\xi\eta} = 0}$$

☆☆ If $F=0$, skip to this step!

Integrate wrt η

$$u_\xi = G(\xi)$$

Integrate wrt ξ

$$u = g(\xi) + h(\eta)$$

antiderivative of G -- we don't need to know G

$$\boxed{u(x,y) = g\left(x - \tfrac{2}{10}y\right) + h(x-y)}$$

switching back to x & y

general solution

$$A u_{xx} + B u_{xt} + C u_{tt} + F(x, t, u, u_x, u_t) = 0$$

$D = B^2 - 4AC$ Parabolic

Transformation:

$$\xi = x \ , \ \tau = x - \frac{B}{2C} t$$

canonical form: $u_{\xi\xi} + H(\xi, \tau, u, u_\xi, u_\tau) = 0$

Note: If $C = 0$ then B must equal 0 since $D = B^2 - 4AC = 0$ in this form. Hopefully our original d.e. isn't too bad with $A u_{xx} + F(---) = 0$

$\boxed{\text{Ex 2}}$ Classify $u_{xx} - 6u_{xy} + 9u_{yy} = xy^2$ as hyperbolic, parabolic, or elliptic & solve the d.e.

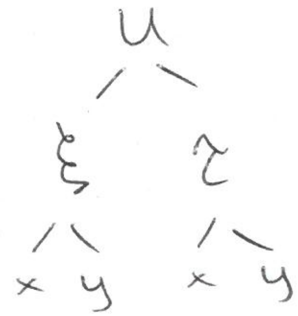
$A = 1 \quad B = -6 \quad C = 9$

$D = B^2 - 4AC = 36 - 36 = 0$ parabolic!

$\xi = x \quad \tau = x - \frac{B}{2C} \cancel{t}y = x + \frac{6}{18} y = x + \frac{1}{3} y$

- If $F(x, y, u, u_x, u_y) = 0$ then we could jump to the canonical form of parabolic with 0 plugged in for H. The issue here is that we don't know the coefficient of $u_{\xi\xi}$:(

Chain rule



$$u_x = u_\xi \xi_x + u_\tau \tau_x$$
$$= u_\xi 1 + u_\tau 1$$

$$u_{xx} = [u_x]_\xi \xi_x + [u_x]_\tau \tau_x$$
$$= [u_\xi + u_\tau]_\xi 1 + [u_\xi + u_\tau]_\tau 1$$
$$= u_{\xi\xi} + \underbrace{u_{\tau\xi} + u_{\xi\tau}}_{2u_{\xi\tau}} + u_{\tau\tau}$$

Clairaut's thm!

$$u_{xy} = \underbrace{[u_x]_\xi}_{\text{we just found this!}} \xi_y + [u_x]_\tau \tau_y$$

$$= (u_{\xi\xi} + u_{\tau\xi})\, 0 + (u_{\xi\tau} + u_{\tau\tau}) \tfrac{1}{3}$$

$\xi = x$ so $\xi_y = 0$

$$u_y = u_\xi \overset{0}{\cancel{\xi_y}} + u_\tau \tau_y$$
$$= \tfrac{1}{3} u_\tau$$

$$u_{yy} = \tfrac{1}{3}[u_y]_\tau = \tfrac{1}{3}\left[\tfrac{1}{3}u_\tau\right]_\tau = \tfrac{1}{9} u_{\tau\tau}$$

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$$u_{xx} - 6u_{xy} + 9u_{yy} = xy^2$$

$$\underbrace{\left(u_{\xi\xi} + 2u_{\xi\tau} + u_{\tau\tau}\right)}_{u_{xx} \text{ from previous page}} - 6\left(\tfrac{1}{3}\left(u_{\xi\tau} + u_{\tau\tau}\right)\right) + 9\left(\tfrac{1}{9}u_{\tau\tau}\right)$$

$$= \underbrace{xy^2}$$

We'll mess around with this after we clean up the left side.

$$u_{\xi\xi} + \cancel{2u_{\xi\tau}} + \cancel{u_{\tau\tau}} - \cancel{2u_{\xi\tau}} - \cancel{2u_{\tau\tau}} + \cancel{u_{\tau\tau}} = xy^2$$

We actually aren't guaranteed that this coefficient is 1 -- ok to feel annoyed still!

$$u_{\xi\xi} = xy^2$$

$\xi = x \quad \tau = x + \frac{1}{3}y \rightarrow 3\tau = 3x + y$

$x = \xi$

$y = 3\tau - 3x = 3\tau - 3\xi$

$$u_{\xi\xi} = \underbrace{\xi}_{x}\underbrace{(3\tau - 3\xi)^2}_{y^2}$$

$$= 9\xi\left[\tau^2 - 2\tau\xi + \xi^2\right]$$

$$= 9\xi\tau^2 - 18\tau\xi^2 + 9\xi^3$$

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$$u_{\xi\xi} = 9\xi\tau^2 - 18\tau\xi^2 + 9\xi^3$$

antiderivative wrt ξ

$$u_{\xi} = \frac{9}{2}\xi^2\tau^2 - 6\tau\xi^3 + \frac{9}{4}\xi^4 + F(\tau)$$

partial of any function of just τ wrt to $\xi = 0$

antiderivative wrt ξ

$$u = \frac{3}{2}\xi^3\tau^2 - \frac{6}{4}\tau\xi^4 + \frac{9}{20}\xi^5 + F(\tau)\xi + G(\tau)$$

switch back to x & y

$$\boxed{u(x,y) = \frac{3}{2}x^3\left(x+\frac{1}{3}y\right)^2 - \frac{6}{4}\left(x+\frac{1}{3}y\right)x^4 + \frac{9}{20}x^5 + F\left(x+\frac{1}{3}y\right)x + G\left(x+\frac{1}{3}y\right)}$$

$$Au_{xx} + Bu_{xt} + Cu_{tt} + F(x,t,u,u_x,u_t) = 0$$

$$D = B^2 - 4AC < 0 \quad \underline{\text{elliptic}}!$$

$$\alpha = x - \frac{B}{2C}t \qquad \beta = \frac{\sqrt{D}}{2iC}t$$

Canonical form:

$$u_{\alpha\alpha} + u_{\beta\beta} + K(\alpha,\beta,u,u_\alpha,u_\beta) = 0$$

Ex 3 Use transformations to reduce $4u_{xx} + 4u_{xy} + 5u_{yy} = 1$ to its canonical form

$$D = B^2 - 4AC = 16 - 4\cdot4\cdot5 = 16(1-5) = -64$$

$$\alpha = x - \frac{4}{10}y \qquad \beta = \frac{8i}{2i(5)}y$$

$$\alpha = x - \frac{2}{5}y \qquad \beta = \frac{4}{5}y$$

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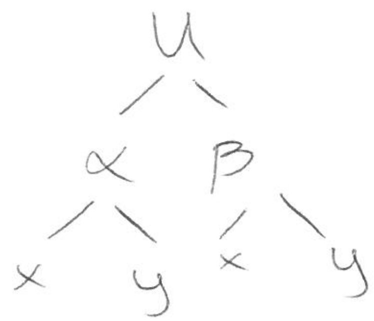

$$u_x = u_\alpha \alpha_x + u_\beta \beta_x$$
$$= u_\alpha 1 + u_\beta \cdot 0$$
$$= u_\alpha \cdot 1$$
$$u_{xx} = [u_x]_\alpha \cdot 1$$
$$= [u_\alpha]_\alpha = u_{\alpha\alpha}$$
$$u_{xy} = [u_x]_\alpha \alpha_y + [u_x]_\beta \beta_y$$
$$= [u_\alpha]_\alpha \left(-\tfrac{2}{5}\right) + [u_\alpha]_\beta \tfrac{4}{5}$$
$$= -\tfrac{2}{5} u_{\alpha\alpha} + \tfrac{4}{5} u_{\alpha\beta}$$

$$u_y = u_\alpha \alpha_y + u_\beta \beta_y$$
$$= u_\alpha \left(\tfrac{-2}{5}\right) + u_\beta \left(\tfrac{4}{5}\right)$$
$$u_{yy} = \left[-\tfrac{2}{5} u_\alpha + \tfrac{4}{5} u_\beta\right]_\alpha \left(-\tfrac{2}{5}\right) +$$
$$\left[-\tfrac{2}{5} u_\alpha + \tfrac{4}{5} u_\beta\right]_\beta \left(\tfrac{4}{5}\right)$$

$$u_{yy} = \tfrac{4}{25} u_{\alpha\alpha} \underbrace{- \tfrac{8}{25} u_{\beta\alpha} - \tfrac{8}{25} u_{\alpha\beta}}_{-\frac{16}{25} u_{\alpha\beta}} + \tfrac{16}{25} u_{\beta\beta}$$

Plugging into

$$4u_{xx} + 4u_{xy} + 5u_{yy} = 1$$
$$4u_{\alpha\alpha} + 4\left(-\tfrac{2}{5} u_{\alpha\alpha} + \tfrac{4}{5} u_{\alpha\beta}\right) + 5\left(\tfrac{4}{25} u_{\alpha\alpha} - \tfrac{16}{25} u_{\alpha\beta} + \tfrac{16}{25} u_{\beta\beta}\right) = 1$$
$$\left(4 - \tfrac{8}{5} + \tfrac{4}{5}\right) u_{\alpha\alpha} + \left(\tfrac{16}{5} - \tfrac{16}{5}\right) u_{\alpha\beta} + \left(\tfrac{16}{5}\right) u_{\beta\beta} = 1$$
$$\tfrac{16}{5} u_{\alpha\alpha} + \tfrac{16}{5} u_{\beta\beta} = 1 \longrightarrow \boxed{u_{\alpha\alpha} + u_{\beta\beta} = \tfrac{5}{16}}$$ ✓

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Examples from <u>Elements of Partial Differential Equations</u> by Drábek & Holubová

$\boxed{\text{Ex 4}}$ Classify $xu_{xx} - 4u_{xt} = 0$ in the domain $x > 0$. Solve by making the nonlinear substitution: $\tau = t$, $\xi = t + 4\ln x$

If $A, B, \& C$ are functions of (x,t) we can still classify them, but we <u>might</u> need help with the transformations

$$D = B^2 - 4AC = (-4)^2 - 4x \cdot 0 = 16 > 0$$

<u>hyperbolic</u>

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$\xi = t + 4\ln x$

$\tau = t$

$$u_x = u_\xi \xi_x + u_\tau \tau_x$$

$$= u_\xi \frac{4}{x} + \cancel{u_\tau \cdot 0}$$

$$u_{xx} = [u_x]_\xi \frac{4}{x} = [\underbrace{u_\xi \tfrac{4}{x}}]_\xi \frac{4}{x} = \left[u_{\xi\xi}\frac{4}{x} + u_\xi [4/x]_\xi\right]\frac{4}{x}$$

this is tricky! since our x is a function of ξ we can't treat this $\frac{4}{x}$ as a constant & we need to do product rule

$$u_{xt} = [u_x]_\xi \xi_t + [u_x]_\tau \tau_t$$

$$= \left[u_{\xi\xi}\frac{4}{x} + u_\xi \underbrace{\left[\tfrac{4}{x}\right]_\xi}\right]\cdot 1 + \left[\underbrace{u_\xi \tfrac{4}{x}}\right]_\tau \cdot 1$$

another product rule needed here!

holding off on finding this hoping it cancels out

$$= u_{\xi\xi}\frac{4}{x} + u_\xi \left[\frac{4}{x}\right]_\xi + u_{\xi\tau}\frac{4}{x} + u_\xi\left[\frac{4}{x}\right]_\tau$$

Plugging back into our d.e:

$$x u_{xx} - 4u_{xt} = 0$$

$$\cancel{\frac{16}{x}u_{\xi\xi}} + \cancel{4u_\xi [4/x]_\xi} - 4\left(\cancel{u_{\xi\xi}\frac{4}{x}} + \cancel{u_\xi\left[\frac{4}{x}\right]_\xi} + u_{\xi\tau}\frac{4}{x} + u_\xi\left[\frac{4}{x}\right]_\tau\right) = 0$$

$$-4\left(u_{\xi\tau}\frac{4}{x} + u_\xi\left[\frac{4}{x}\right]_\tau\right) = 0$$

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$\xi = t + 4\ln x$ need to solve for x

$\xi - t = 4\ln x$

$\frac{\xi - t}{4} = \ln x$

$\tau = t \rightarrow \frac{\xi - \tau}{4} = \ln x$

$x = e^{\frac{\xi-\tau}{4}} = e^{\xi/4} e^{-\tau/4}$ so $\frac{4}{x} = 4\underbrace{e^{-\xi/4} e^{\tau/4}}_{\frac{1}{x}}$

$-4\left(u_{\xi\tau}\frac{4}{x} + u_\xi \left[4e^{-\xi/4}e^{\tau/4}\right]_\tau\right) = 0$

$-4\left(u_{\xi\tau}\frac{4}{x} + u_\xi \left(\cancel{4}\cdot\cancel{\tfrac{1}{4}}\underbrace{e^{-\xi/4}e^{\tau/4}}_{\frac{1}{x}}\right)\right) = 0$

$\frac{4}{x}u_{\xi\tau} + \frac{1}{x}u_\xi = 0$

$4u_{\xi\tau} + u_\xi = 0$

$v = u_\xi \rightarrow 4v_\tau + v = 0$

☆ If we tried $v = u_x$ at the beginning of the problem, we'd end up with $u_t = F(\xi)$ & a bit stuck ☆

$4v_\tau + v = 0 \rightarrow v_\tau = -\frac{1}{4}v$

$\int \frac{\partial v}{v} = \int -\frac{1}{4}\,\partial\tau$

$$\ln|v| = -\frac{1}{4}\tau + G(\xi)$$

$$v = \underbrace{H(\xi)}_{\pm e^{G(\xi)}} e^{-\tau/4}$$

$v = u_\xi$ so

$$u_\xi = H(\xi)\, e^{-\tau/4}$$

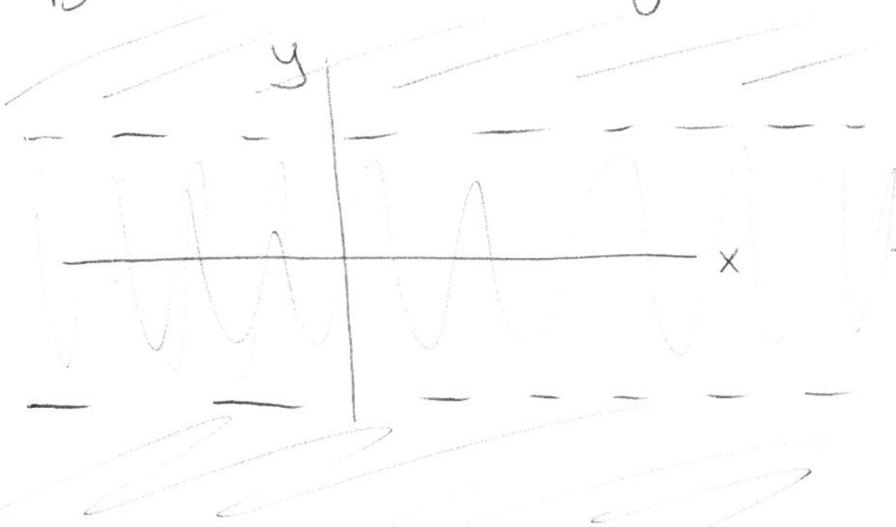
$$u = h(\xi)\, e^{-\tau/4} + f(\tau)$$

$$\boxed{u(x,t) = h(t + 4\ln x)\, e^{-t/4} + f(t)}$$

Ex 5 Classify the PDE & if necessary give qualifications for when it is elliptic, hyperbolia or parabolic in the xy-plane

$$u_{xx} + 2y\,u_{xy} + u_{yy} + u = 0$$

$$D = B^2 - 4AC = 4y^2 - 4 = 4(y^2 - 1)$$



$y > 1$, $D > 0$ Hyperbolic

$y = 1$ parabolic, $D = 0$

$-1 < y < 1$, $D < 0$, elliptic

$y = -1$ parabolic, $D = 0$

$y < -1$, $D > 0$ Hyperbolic