11 Method of Characteristics

Useful for solving:

(★) $u_t + c(x,t)u_x = f(x,t,u)$

Technique:

1. If the coefficient of u_t is not 1, divide the equation by that coefficient to get (★)

2. Set $\frac{dx}{dt} = c(x,t)$, the coefficient of u_x in (★)

3. Solve $\frac{dx}{dt} = c(x,t)$ & solve that solution for C_1 in terms of x & t

4. Set $\xi = C_1$ & $\tau = t$

5. Since $\frac{dx}{dt} = c(x,t)$, when we use the chain rule to find u_t & u_x for $u(\xi,\tau)$ & plug into (★) we'll get $u_\tau = f$ (put f in terms of u, τ, & ξ) & solve for $u(\xi,\tau)$ & switch back to $u(x,t)$

6. Use given conditions to find a specific solution if possible

21 Examples from Applied Partial Differential Equations by DuChateau & Zachmann

Use the method of characteristics to solve the following initial value problems

$\boxed{\text{Ex 1}}$ $u_t + u_x = -u \ , \ u(x,0) = 1 + \cos x$

Note! It is already in the form

$u_t + c(x,t)u_x = f(x,t,u)$

Set $\frac{dx}{dt} = c(x,t)$

$\frac{dx}{dt} = 1$

$\int dx = \int dt$

$x = t + C_1$

Treating this as a separable since that comes up often. See the worksheet on separable equations if you need detailed practice.

Solve for C_1:

$C_1 = x - t$

$\xi = C_1 = x - t \ , \ \tau = t$

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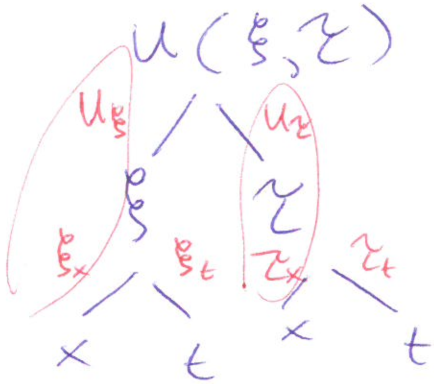
Chain Rule from Calc 3 (see worksheet on chain rule if you need more help!)



Tree Diagram (use one of these if you find it helpful)

For this example: $\xi = x-t$ & $\tau = t$

so $\xi_x = 1, \xi_t = -1$

$\tau_x = 0, \tau_t = 1$

$$u_x = u_\xi \xi_x + u_\tau \tau_x$$

$$u_x = u_\xi 1 + u_\tau 0$$

$$u_x = u_\xi$$

$$\left.u_t = u_\xi \xi_t + u_\tau \tau_t\right\} \text{chain rule}$$

$$= u_\xi(-1) + u_\tau 1$$

Plug into $u_t + \underset{c(x,t)}{1}u_x = \underbrace{-u}_{f}$ (the differential eqn they gave us at the beginning)

$$\left(u_\xi(-1) + u_\tau\right) + \left(u_\xi\right) = -u$$

$$u_\tau = -u$$

Note: We're getting $u_\tau = f$ a PDE with only partials wrt one variable τ. We can solve this like an ODE, but instead of a "+ C" we'll get "$+F(\xi)$"

variable we're not differentiating wrt in $u_\tau = f$

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$u_\tau = -u$ Treat it like a Separable!

$$\frac{du}{d\tau} = -u$$

$$\int \frac{\partial u}{u} = \int -\partial \tau$$

$$\ln|u| = -\tau + F(\xi)$$

$$|u| = e^{-\tau + F(\xi)} = e^{-\tau} e^{F(\xi)}$$

$$u = G(\xi) e^{-\tau}$$

similar to when we do

$$\frac{dy}{dx} = y$$

$$\int \frac{dy}{y} = \int 1\, dx$$

$$\ln|y| = x + C$$

$$|y| = e^{x + C} = e^{x} e^{C}$$

$$y = Ke^{x}, \quad K = \pm e^{C}$$

$$u(\xi, \tau) = G(\xi) e^{-\tau}$$

☆ switch back to x & t:

$$\xi = x - t, \quad \tau = t$$

$$u(x,t) = G(x-t) e^{-t}$$

☆ Plug into initial condition:

$$u(x,0) = 1 + \cos x = G(x-0) e^{-0} = G(x) 1 = G(x)$$

$$G(x) = 1 + \cos x$$

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$$u(x,t) = G(x-t)e^{-t}$$

$G(x) = 1 + \cos x$ so

$G(x-t) = 1 + \cos(x-t)$ Precalc skills!

so $\boxed{u(x,t) = (1+\cos(x-t))e^{-t}}$

$\boxed{\text{Ex 2}}$ $u_x + x u_t = u^2$

$u(x,0) = 1, \quad x > 0$

Note: This is not in the correct form. We want $u_t + c(x,t)\, u_x = f(x,t,u)$

Rearranging our differential equation:

$$x u_t + u_x = u^2$$

We need the coefficient in front of $u_t = 1$. Divide both sides by x to get

$$u_t + \underbrace{\frac{1}{x}}_{c(x,t)} u_x = \frac{u^2}{x}$$

$$\frac{dx}{dt} = c(x,t) = \frac{1}{x}$$

6) $\int x\,dx = \int 1\,dt$ Separable!

$\frac{1}{2}x^2 = t + C_1$

$x^2 = 2t + C_2$

the naming of the constant doesn't especially matter but $2C_1$ is just some other constant

Recall we are trying to solve this for the constant & not for x

$C_2 = x^2 - 2t$ so $\xi = C_2 = \underline{x^2 - 2t}$, $\tau = t$

ξ_x

Chain rule of $u(\xi, \tau)$:

$u_x = \xi_x u_\xi + u_\tau \tau_x$

$u_x = u_\xi (2x) + u_\tau (0)$

$u_x = u_\xi (2x)$

$u_t = u_\xi \xi_t + u_\tau \tau_t$

$u_t = u_\xi (-2) + u_\tau 1$

Plugging into our adjusted d.e.:

$u_t + \frac{1}{x} u_x = \frac{u^2}{x}$

$\underbrace{(u_\xi(-2) + u_\tau)}_{u_t} + \frac{1}{x}\underbrace{(u_\xi(2x))}_{u_x} = \frac{u^2}{x}$

$u_\tau = \frac{u^2}{x}$ or $u_\tau = f$!

7)

$$U_\tau = \frac{u^2}{x}$$

f

Need to only have $u, \tau,$ & ξ here.

$\xi = x^2 - 2t \rightarrow x = \sqrt{\xi + 2t}$ (we were told $x>0$)

$\tau = t \qquad x = \sqrt{\xi + 2\tau}$

$$\frac{\partial u}{\partial \tau} = U_\tau = \frac{u^2}{\sqrt{\xi + 2\tau}} \qquad \text{Separable!}$$

$$\int \frac{du}{u^2} = \int \frac{1}{\sqrt{\xi + 2\tau}} \partial \tau$$

$$-\frac{1}{u} = \int \frac{1}{2} (w)^{-1/2} \partial w$$

$$-\frac{1}{u} = w^{1/2} + F(\xi)$$

$$-\frac{1}{u} = \sqrt{\xi + 2\tau} + F(\xi)$$

* Usually I would do a u-sub here, but since we already have u in the problem, I'll use a w

$w = \xi + 2\tau$

$\partial w = 2 \partial \tau$

$\frac{1}{2} \partial w = \partial \tau$

$$u = \frac{-1}{\sqrt{\xi + 2\tau} + F(\xi)}$$

Plug in $\xi = x^2 - 2t$ & $\tau = t$

$$u(x,t) = \frac{-1}{\sqrt{x^2 \cancel{-2t} \cancel{+2t}} + F(x^2 - 2t)} = \frac{-1}{x + F(x^2 - 2t)}$$

IC: $u(x,0) = 1 = \dfrac{-1}{x + F(x^2 - 2 \cdot 0)} = \dfrac{-1}{x + F(x^2)}$

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$$1 = \frac{-1}{x+F(x^2)}$$

$$x+F(x^2) = -1$$

$$F(x^2) = -1-x \to F(x) = -1-\sqrt{x} \to$$

$$F(x^2-2t) = -1-\sqrt{x^2-2t}$$

$$u(x,t) = \frac{-1}{x+F(x^2-2t)} = \boxed{u(x,t) = \frac{-1}{x-1-\sqrt{x^2-2t}}}$$

$\boxed{\text{Ex 3}}$ $\quad u_x - t u_t = 1+u \; , \; u(0,t) = t$

* The correct form is $u_t + c(x,t)u_x = f(x,t,u)$

$$1u_t - \frac{1}{t}u_x = -\frac{1}{t} - \frac{u}{t}$$

$$\frac{dx}{dt} = c(x,t) = -\frac{1}{t}$$

$$\int dx = \int -\frac{1}{t}\,dt$$

$$x = -\ln|t| + C_1$$

$$C_1 = x + \ln|t|$$

$$\xi = x+\ln|t| \; , \; \tau = t$$

* We saw twice how chain rule leads to $u_\tau = f$. Feel free to verify that on your own :)

$$u_\tau = -\frac{1}{t} - \frac{u}{t}$$

We need to swith the t terms to τ. Recall this result is found by finding u_t & u_x in terms of ξ & τ and plugging into $u_t - \frac{1}{t}u_x = \frac{-1}{t} - \frac{u}{t}$

$$u_\tau = \frac{-1}{\tau} - \frac{u}{\tau}$$

First Order Linear! (see the worksheet if you need a reminder)

$$\frac{du}{d\tau} + \frac{1}{\tau}u = \frac{-1}{\tau}$$

$$\mu = e^{\int \frac{1}{\tau} d\tau} = e^{\ln|\tau|} = \tau$$

(we don't need the absolute value or a constant when finding μ)

$$\frac{\partial}{\partial \tau}\left[\mu u\right] = \tau\left(-\frac{1}{\tau}\right)$$

$$\frac{\partial}{\partial \tau}\left[\tau u\right] = -1$$

$$\tau u = \int -1 \, d\tau = -\tau + F(\xi)$$

$$u = -1 + \frac{F(\xi)}{\tau} \rightarrow u(x,t) = -1 + \frac{F(x+\ln|t|)}{t}$$

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$$\text{BC: } u(0,t) = t = -1 + \frac{F(0+\ln|t|)}{t}$$

$$t^2 + t = F(\ln|t|)$$

$$F(\ln|t|) = t^2 + t \rightarrow$$

$$F(\ln e^t) = e^{2t} + e^t$$

$$F(t) = e^{2t} + e^t$$

$$u(x,t) = -1 + \frac{F(x+\ln|t|)}{t}$$

$$u(x,t) = -1 + \frac{e^{2(x+\ln|t|)} + e^{x+\ln|t|}}{t}$$

$$= -1 + \frac{e^{2x}e^{\ln t^2} + e^x e^{\ln|t|}}{t}$$

$$= -1 + \frac{e^{2x}t^2 + e^x t}{t}$$

$$\boxed{u(x,t) = -1 + e^{2x}t + e^x}$$

$\boxed{\text{Ex 4}}$ $u_t + t u_x = x \quad u(x,0) = x^2$

* Right form! $\frac{dx}{dt} = t$

$$\int dx = \int t \, dt$$

$$x = \tfrac{1}{2}t^2 + C$$

$$C = x - \tfrac{1}{2}t^2$$

$$\xi = x - \tfrac{1}{2}t^2$$

$$\tau = t$$

$u_\tau = f$

$u_\tau = x \quad \left(\xi = x - \tfrac{1}{2}\cancel{t}^2 \atop \text{so } x = \xi + \tfrac{1}{2}\tau^2 \right)$

Plug in for x:

$u_\tau = \xi + \tfrac{1}{2}\tau^2$ Separable!

$$\int du = \int \xi + \tfrac{1}{2}\tau^2 \, d\tau$$

$$u = \xi\tau + \tfrac{1}{6}\tau^3 + F(\xi)$$

$$u(x,t) = (x - \tfrac{1}{2}t^2)t + \tfrac{1}{6}t^3 + F(x - \tfrac{1}{2}t^2)$$

IC: $u(x,0) = (x - \tfrac{1}{2}0^2)\cdot 0 + \tfrac{1}{6}\cdot 0^3 + F(x - \tfrac{1}{2}\cdot 0^2) = F(x) = x^2$

$\rightarrow F(x - \tfrac{1}{2}t^2) = (x - \tfrac{1}{2}t^2)^2 \rightarrow \boxed{u(x,t) = (x - \tfrac{1}{2}t^2)t + \tfrac{1}{6}t^3 + (x - \tfrac{1}{2}t^2)^2}$