Test 1 Problem Solutions

☆ Questions in a separate pdf

$\boxed{\text{Ex 1}}$ $y\,u_{xy} + 2u_x = x$

$v = u_x$

$y\,v_y + 2v = x$

1st order lin:

$$v_y + \frac{2}{y}v = \frac{x}{y}$$

$$\mu(y) = e^{\int \frac{2}{y}dy} = e^{2\ln y} = e^{\ln y^2} = y^2$$

$$\frac{\partial}{\partial y}\left(y^2 v\right) = \frac{x}{y}y^2 = xy$$

$$y^2 v = \int xy\,dy$$

$$= \frac{1}{2}xy^2 + G(x)$$

$$v = \frac{\frac{1}{2}xy^2 + G(x)}{y^2}$$

$$= \frac{1}{2}x + \frac{G(x)}{y^2}$$

$$u_x = \frac{1}{2}x + \frac{G(x)}{y^2}$$

$$u = \frac{1}{4}x^2 + \frac{g(x)}{y^2} + h(y)$$

2) $u(x,1) = 0 = \frac{1}{4}x^2 + \frac{g(x)}{1^2} + h(1)$

$$g(x) = -\frac{1}{4}x^2 - h(1)$$

$$u(0,y) = 0 = \frac{1}{4}\cdot 0^2 + \frac{g(0)}{y^2} + h(y)$$

$$h(y) = \frac{-g(0)}{y^2} = \frac{c}{y^2} \leftarrow \text{some constant}$$

$$u(x,y) = \frac{1}{4}x^2 + \frac{g(x)}{y^2} + h(y)$$

$$= \frac{1}{4}x^2 + \frac{\left(-\frac{1}{4}x^2 - h(1)\right)}{y^2} + \frac{c}{y^2}$$

$$= \frac{1}{4}x^2 - \frac{1}{4}\frac{x^2}{y^2} - \frac{h(1)}{y^2} + \frac{c}{y^2}$$

$h(1) = \frac{c}{1^2} = c$ so

$$u(x,y) = \frac{1}{4}x^2 - \frac{1}{4}\frac{x^2}{y^2} - \frac{c}{y^2} + \frac{c}{y^2}$$

$$u(x,y) = \frac{1}{4}x^2 - \frac{1}{4}\frac{x^2}{y^2}$$

$\boxed{\text{Ex 2}}$ $u_x - 2u_t = 0$ Characteristics

want this to be a 1

$$u_t + \left(-\tfrac{1}{2}\right)u_x = -\tfrac{1}{2}\cdot 0 = 0$$

$$\frac{dx}{dt} = -\frac{1}{2}$$

$$\int dx = \int -\tfrac{1}{2}\,dt$$

$$x = -\tfrac{1}{2}t + C$$

$$x + \frac{t}{2} = C = \xi$$

$$u_\xi = 0$$

$$u = g(\xi)$$

$$u(x,t) = g\left(x + \frac{t}{2}\right)$$

$$u(x,e^x) = \underbrace{e^{2x} + 4xe^x + 4x^2}_{\text{this factors!}} = g\left(x + \frac{e^x}{2}\right)$$

$$\left(e^x + 2x\right)^2 = g\left(x + \frac{e^x}{2}\right)$$

$$\left(2\left(\frac{e^x}{2} + x\right)\right)^2 = g\left(x + \frac{e^x}{2}\right)$$

$$4\left(\frac{e^x}{2} + x\right)^2 = g\left(x + \frac{e^x}{2}\right)$$

so $g(v) = 4v^2$ so

$$\boxed{u(x,t) = 4\left(x + \frac{t}{2}\right)^2}$$

$\boxed{Ex\,3}$

$$xu_x - xt\,u_t = 0$$ Method of characteristics

$$u_t + \frac{xu_x}{-xt} = \frac{0}{-xt}$$ if this side isn't 0

$$u_t - \frac{u_x}{t} = 0$$

$$\frac{dx}{dt} = -\frac{1}{t}$$

$$\int dx = \int -\frac{1}{t}\,dt$$

$$x = -\ln|t| + C$$

$$\xi = C = x + \ln|t|$$

$$u_\tau = 0$$

$$u = g(\xi)$$

$$u(x,t) = g(x + \ln|t|)$$

$$u(x,x) = x^2 e^{2x} = g(x + \ln|x|)$$

$$x + \ln|x|$$
$$= \ln|e^x| + \ln|x|$$ $(\ln(ab) = \ln a + \ln b)$
$$= \ln(\underbrace{|e^x x|}_{v})$$

$$g(\ln|v|) = x^2 e^{2x} = (xe^x)^2 = v^2$$

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$w = \ln|v|$

$e^{w} = |v|$

$v^2 = e^{2w}$

$u(x,x) = g(x + \ln|x|)$
$= g(\ln|e^x x|)$
$g(w) = e^{2w}$

$u(x,t) = g(x + \ln|t|)$
$= e^{2(x+\ln|t|)} = e^{2x} e^{2\ln|t|} = e^{2x} t^2$

$$\boxed{u(x,t) = t^2 e^{2x}}$$

Ex 4

$u = x^2 + y^2$
$u_x = 2x \quad u_y = 2y$

$y u_x - x u_y = y(2x) - x(2y) = 0$ ✓ yes

Ex 5

$$u_{xx} + 2u_{xt} + u_{tt} + u_x + u_t = 0$$

$D = B^2 - 4AC = 4 - 4 = 0 \rightarrow$ parabolic

$\xi = x \quad \tau = x - \frac{B}{2C} t = x - \frac{2}{2} t = x - t$

$u_x = u_\xi \xi_x + u_\tau \tau_x$
$= u_\xi 1 + u_\tau \cdot 1$

$u_{xx} = [u_\xi + u_\tau]_\xi \cdot 1 + [u_\xi + u_\tau]_\tau \cdot 1$
$= u_{\xi\xi} + u_{\tau\xi} + u_{\xi\tau} + u_{\tau\tau}$

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$$u_t = u_\xi \xi_t + u_\tau \tau_t = u_\xi \cdot 0 + u_\tau(-1) = -u_\tau$$

$$u_{xt} = -[\underbrace{u_\xi \cdot 1 + u_\tau \cdot 1}_{u_x}]_\tau$$

$$= -u_{\xi\tau} - u_{\tau\tau}$$

$$u_{tt} = -[-u_\tau]_\tau = u_{\tau\tau}$$

$$u_{xx} + 2u_{xt} + u_{tt} + \underbrace{u_x + u_t}_{} = 0$$

F ≠ 0 which is why we are doing so much chain rule.

Plug in

$$u_{\xi\xi} + \cancel{2u_{\tau\xi}} + u_{\tau\tau} + 2(\cancel{-u_{\xi\tau}} - u_{\tau\tau}) + u_{\tau\tau} + u_\xi + \cancel{u_\tau} \cancel{-u_\tau} = 0$$

$$u_{\xi\xi} + u_\xi = 0$$

$$u_{\xi\xi} = -u_\xi$$

$$u_\xi = -u + f(\tau)$$

This can be separable or 1st order linear

$$\frac{\partial u}{\partial \xi} = -(u - f(\tau))$$

$$\int \frac{\partial u}{u - f(\tau)} = \int \partial \xi$$

$$\ln|u - f(\tau)| = -\xi + g(\tau)$$

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$$|u - f(\tau)| = e^{-\xi} e^{g(\tau)}$$

$$u - f(\tau) = h(\tau) e^{-\xi}$$

$$u = f(\tau) + h(\tau) e^{-\xi}$$

$$\boxed{u(x,t) = f(x-t) + h(x-t)\, e^{-x}}$$

$\boxed{\text{Ex } 6}$ $\quad 2u_{xx} - 4u_{xy} - 6u_{yy} + u_x = 0$

$$D = B^2 - 4AC = 16 - 4(2)(-6)$$
$$= 16 + 48$$
$$= 64 > 0$$

hyperbolic

$$\xi = x + \left(\frac{-B + \sqrt{D}}{2C}\right) y \qquad \tau = x + \left(\frac{-B - \sqrt{D}}{2C}\right) y$$

$$= x + \left(\frac{4 + \sqrt{64}}{-12}\right) y \qquad = x + \left(\frac{4 - \sqrt{64}}{-12}\right) y$$

$$= x - y \qquad = x + \tfrac{1}{3} y$$

$$u_x = u_\xi \xi_x + u_\tau \tau_x$$
$$= u_\xi 1 + u_\tau 1$$
$$u_{xx} = [u_\xi + u_\tau]_\xi \cdot 1 + [u_\xi + u_\tau]_\tau \cdot 1$$
$$= u_{\xi\xi} + u_{\tau\xi} + u_{\xi\tau} + u_{\tau\tau}$$
$$u_y = u_\xi \xi_y + u_\tau \tau_y = u_\xi(-1) + u_\tau\left(\tfrac{1}{3}\right)$$

8)

$$u_{xy} = \left[u_\xi \cdot 1 + u_\tau \cdot 1\right]_\xi (-1) + \left[u_\xi \cdot 1 + u_\tau \cdot 1\right]_\tau \left(\tfrac{1}{3}\right)$$

$$= (u_{\xi\xi} + u_{\tau\xi})(-1) \quad + \tfrac{1}{3} u_{\xi\tau} + \tfrac{1}{3} u_{\tau\tau}$$

$$u_{yy} = \left[u_\xi(-1) + u_\tau \tfrac{1}{3}\right]_\xi (-1) + \left[u_\xi(-1) + u_\tau\left(\tfrac{1}{3}\right)\right]_\tau \tfrac{1}{3}$$

$$= u_{\xi\xi} - \tfrac{1}{3} u_{\tau\xi} - \tfrac{1}{3} u_{\xi\tau} + \tfrac{1}{9} u_{\tau\tau}$$

$$2u_{xx} - 4u_{xy} - 6u_{yy} + u_x = 0$$

$$2\left(u_{\xi\xi} + 2_{\xi\tau} + u_{\tau\tau}\right) - 4\left(-u_{\xi\xi} - \tfrac{2}{3} u_{\xi\tau} + \tfrac{1}{3} u_{\tau\tau}\right)$$

$$- 6\left(u_{\xi\xi} - \tfrac{2}{3} u_{\xi\tau} + \tfrac{1}{9} u_{\tau\tau}\right) + u_\xi + u_\tau = 0$$

$$4 + \tfrac{8}{3} + \tfrac{12}{3} = \tfrac{32}{3} \quad u_{\xi\tau}$$

$$\left.2 - \tfrac{4}{3} - \tfrac{6}{9} = 2 - \tfrac{4}{3} - \tfrac{2}{3} = 2 - \tfrac{6}{3} = 0\right\} \quad u_{\tau\tau}$$

$$\boxed{\tfrac{32}{3} u_{\xi\tau} + u_\xi + u_\tau = 0}$$

$\boxed{\text{Ex 7}}$ $D = B^2 - 4AC = 16 - 4(2)(-6)$
$= 16 + 48 = 64 > 0$
hyperbolic

$F = 0$ so

$\xi = x - y \quad \eta = x + \frac{1}{3}y$ } This is the same as the last ex but $F=0$

Canonical form when $F = 0$ is

$$u_{\xi\eta} = 0$$
$$u_\xi = G(\xi)$$
$$u = g(\xi) + h(\eta)$$
$$\boxed{u(x,t) = g(x-y) + h\left(x + \tfrac{1}{3}y\right)}$$

$\boxed{\text{Ex 8}}$ $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = \underbrace{r^2}_{x^2+y^2}$

$u(x,y) = f(x^2+y^2) \rightarrow u(r,\theta) = f(r^2) \rightarrow$ no dependence on θ so $u_{\theta\theta} = 0$

$$u_{rr} + \frac{1}{r}u_r + \cancel{\frac{1}{r^2}0} = r^2$$

Let $V = u_r \rightarrow V_r + \frac{1}{r}V = r^2$

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$V_r + \frac{1}{r}V = r^2$

$\mu(r) = e^{\int \frac{1}{r}dr} = e^{\ln r} = r$

$\frac{\partial}{\partial r}[rV] = r^3$

$rV = \frac{1}{4}r^4 + C$

recall: V just depends on θ

$V = \frac{1}{4}r^3 + \frac{C}{r}$

$U_r = \frac{1}{4}r^3 + \frac{C}{r}$

$U = \frac{1}{16}r^4 + C\ln r + D$

$r > 0$, omit abs val if you want

$U(x,y) = 0$ when $x^2+y^2=1$

$U(r,\theta) = 0$ when $r=1$

$(r^2)^2$

$U(1,\theta) = 0 = \frac{1}{16}\cdot 1^2 + \cancel{C\ln 1} + D$

$D = -\frac{1}{16}$

$U(r,\theta) = \frac{1}{16}r^4 + C\ln r - \frac{1}{16}$

as $r \to 0^+$ $\ln r \to -\infty$ recall maxs/mins should happen on boundary, choose $C=0$

$U(r,\theta) = \frac{1}{16}r^4 - \frac{1}{16} \to U(x,y) = \frac{1}{16}(x^2+y^2)^2 - \frac{1}{16}$

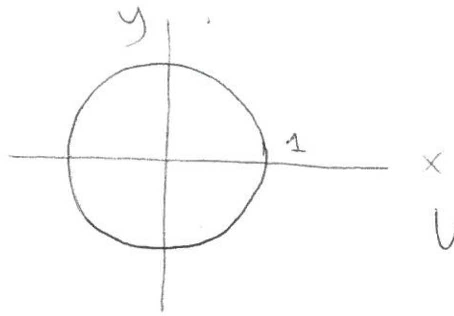
$\boxed{\text{Ex 9}}$ Solutions to $\Delta u = 0$ have their max & min values on the boundary

$u(x,y) = 4x^3$

$u_x(x,y) = 12x^2 = 0$ if $x = 0$ $\quad u(0,y) = 4 \cdot 0^3 = 0$

not the max or min

we can observe this by testing other points on the boundary



$u(1,y) = 4 \cdot 1^3 = \boxed{4}$

biggest value of x on boundary

max

Didn't ask but min: $u(-1,y) = -4$

smallest value on boundary

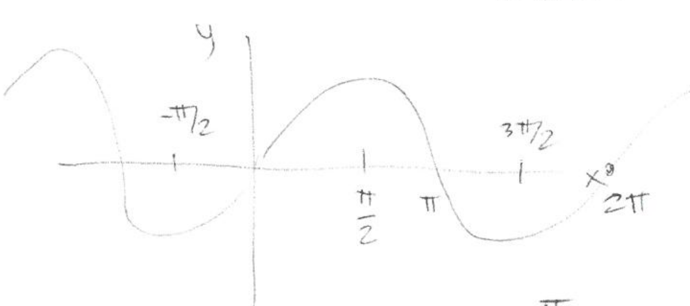
$\boxed{\text{Ex 10}}$ a) max & mins of Laplace's eqn occur on the boundary

$u(1,\theta) = 2 + 3\sin\theta$

$-1 \leq \sin\theta \leq 1$

max value when $\sin\theta = 1 \rightarrow \theta = \frac{\pi}{2}$

$\sin\theta = -1 \rightarrow \theta = \frac{3\pi}{2}$



$u(1, \frac{\pi}{2}) = 2 + 3\sin\frac{\pi}{2} = 5 = \text{max}$

$u(1, \frac{3\pi}{2}) = 2 + 3\sin\frac{3\pi}{2}) = 2 - 3 = -1 = \text{min}$

b) $u(0,0) = \frac{1}{2\pi 1} \int_0^{2\pi} (2 + 3\sin\theta)\, 1\, d\theta$

$= \frac{1}{2\pi}\left[2\theta - 3\cos\theta\right]_0^{2\pi} = 2$

c) No, unless $u(x,y) = \text{Constant}$ it doesn't attain its max on the interior