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Fourier Transforms

Technique:
1. Fourier transform
2. Solve the "ODE"
3. Inverse transform

Note: different books use different definitions of the Fourier transform!!

Ex 1

$u_{tt} = u_{xx}$, $\underbrace{-\infty < x < \infty}$, $t > 0$

Fourier transforms are frequently $-\infty < x < \infty$
Laplace $0 < x < \infty$

$u(x,0) = \frac{1}{1+x^2}$

$u_t(x,0) = 0$

$\mathcal{F}\{u_{tt}\} = \mathcal{F}\{u_{xx}\}$

$\hat{u}_{tt}(\xi, t) = (-i\xi)^2 \hat{u}(\xi, t)$ ts are unaffected

$\hat{u}_{tt} = -\xi^2 \hat{u}$

$\hat{u}_{tt} + \xi^2 \hat{u} = 0$

$r^2 + \xi^2 = 0$

$r = \pm\sqrt{-\xi^2} = \pm \xi i$

Recall:
$ay'' + by' + cy = 0$
$ar^2 + br + c = 0$
If $b^2 - 4ac < 0$ we get
$r = \alpha \pm \beta i$
$y = e^{\alpha t}\left[c_1 \cos\beta t + c_2 \sin\beta t\right]$

$\alpha = 0$, $\beta = \xi$

$\hat{u}(\xi, t) = c_1(\xi)\cos\xi t + c_2(\xi)\sin\xi t$

Unfortunately, cos & sin are bounded so we can't just set c_1 or c_2 to 0.

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$$\hat{u}(\xi,t) = C_1(\xi)\cos(\xi t) + C_2(\xi)\sin(\xi t)$$

$$\mathcal{F}\left\{u(x,0) = \frac{1}{1+x^2}\right\} \rightarrow \hat{u}(\xi,0) = \frac{\pi}{1} e^{-1|\xi|}$$
$$= \pi e^{-1|\xi|}$$

$$\mathcal{F}\left\{\frac{a}{\pi}\frac{1}{x^2+a^2}\right\} = e^{-a|\xi|}$$

our $a=1$

$$\hat{u}(\xi,0) = C_1(\xi)\cos 0 + \cancel{C_2(\xi)\sin 0}$$
$$= C_1(\xi) = \pi e^{-1|\xi|}$$

$$\hat{u}(\xi,t) = \pi e^{-1|\xi|}\cos(\xi t) + C_2(\xi)\sin(\xi t)$$

$$\hat{u}_t(\xi,t) = -\xi\pi e^{-1|\xi|}\sin(\xi,t) + \xi C_2(\xi)\cos(\xi t)$$

$$\hat{u}_t(\xi,0) = -\xi\pi e^{-1|\xi|}\cancel{\sin 0} + \xi C_2(\xi)\cos 0$$
$$= \xi C_2(\xi)$$

$$\mathcal{F}\{u_t(x,0) = 0\} \longrightarrow \hat{u}_t(\xi,0) = 0$$

$\hat{u}_t(\xi,0) = \xi C_2(\xi) = 0$ for all $\xi \rightarrow C_2 = 0$

$$\hat{u}(\xi,t) = \underbrace{\pi e^{-1|\xi|}}_{\hat{f}}\underbrace{\cos(\xi t)}_{\hat{g}}$$

$$\mathcal{F}\{f * g\} = \hat{f}(\xi)\hat{g}(\xi)$$ Fourier of a convolution is a product

Recall: $u * v = \int_{-\infty}^{\infty} u(y)\, v(x-y)\, dy$

so $\hat{u}(\xi, t) = \pi e^{-|\xi|} \cos(\xi t)$

$$u(x,t) = \mathcal{F}^{-1}\{\pi e^{-|\xi|}\} * \mathcal{F}^{-1}\{\cos(\xi t)\}$$

$$= \frac{1}{1+x^2} * \underbrace{\mathcal{F}^{-1}\{\cos(\xi t)\}}_{\text{not in our table so new plan}}$$

Recall: $\mathcal{F}^{-1}\{\hat{u}(\xi)\} = u(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{u}(\xi) e^{-i\xi x}\, d\xi$

$$u(x,t) = \mathcal{F}^{-1}\{\pi e^{-|\xi|} \cos(\xi t)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \pi e^{-|\xi|} \cos(\xi t) e^{-i\xi x}\, d\xi$$

Ans: $$\boxed{u(x,t) = \frac{1}{2} \int_{\infty}^{\infty} e^{-|\xi|} \cos(\xi t) e^{-i\xi x}\, d\xi}$$

Frequently we get an integral for our answer. If I asked a problem like this, I'd give you the inverse transform

Ex 2

$$u_t = \frac{1}{4} u_{xx} \qquad -\infty < x < \infty,\ t > 0$$

$$u(x,0) = e^{-x^2}$$

$$\mathcal{F}\{u_t\} = \frac{1}{4} \mathcal{F}\{u_{xx}\}$$

$$\hat{u}_t(\xi, t) = \frac{1}{4}(-i\xi)^2 \hat{u}(\xi, t)$$

$$\hat{u}_t = -\frac{1}{4}\xi^2 \hat{u}$$

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$$\hat{u}_t = -\frac{1}{4}\xi^2 \hat{u}$$

Separable!

$$\int \frac{\partial \hat{u}}{\hat{u}} = \int -\frac{1}{4}\xi^2 \, \partial t$$

$$\ln|\hat{u}| = -\frac{1}{4}\xi^2 t + C_1(\xi)$$

$$|\hat{u}| = e^{-1/4\,\xi^2 t + C_1(\xi)}$$

$$\hat{u} = C_2(\xi) e^{-\frac{\xi^2 t}{4}}$$

$$\mathcal{F}\{u(x,0) = e^{-x^2}\} \longrightarrow \hat{u}(\xi,0) = \sqrt{\pi}\, e^{-\xi^2/4}$$

$$\mathcal{F}\{e^{-ax^2}\} = \sqrt{\frac{\pi}{a}}\, e^{-\left(\xi^2/4a\right)}$$

$$a = 1$$

$$\hat{u}(\xi,0) = C_2(\xi) e^0 = \sqrt{\pi}\, e^{-\xi^2/4}$$

$$\hat{u}(\xi,t) = \sqrt{\pi}\, e^{-\xi^2/4}\, e^{-\xi^2 t/4} = \sqrt{\pi}\, e^{-\frac{\xi^2}{4}(t+1)}$$

we don't have $\sqrt{\frac{\pi}{a}}$ Only $\sqrt{\pi}$

$$\mathcal{F}\{e^{-ax^2}\} = \sqrt{\frac{\pi}{a}}\, e^{-\left(\xi^2/4a\right)}$$

$$\frac{1}{a} = t+1$$

$$a = \frac{1}{t+1}$$

$$u(x,t) = \sqrt{a}\, e^{-ax^2}$$

Ans: $\boxed{u(x,t) = \frac{1}{\sqrt{t+1}}\, e^{-\frac{x^2}{t+1}}}$

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This is an example we did in class but as $u_t = k u_{xx}$

Ex 3 $u_t = u_{xx}$, $-\infty < x < \infty$, $t > 0$

$u(x,0) = f(x)$

$$\mathcal{F}\{u_t\} = \mathcal{F}\{u_{xx}\}$$

$$\hat{u}_t(\xi, t) = -\xi^2 \hat{u}(\xi, t)$$

We solved this pretty much in the last example -- show the steps on a test

$$\hat{u} = c_2(\xi) e^{-\xi^2 t}$$

$$\mathcal{F}\{u(x,0) = f(x)\} \rightarrow \hat{u}(\xi, 0) = \hat{f}(\xi)$$

$$\hat{u}(\xi, 0) = \hat{f}(\xi) = c_2(\xi) e^0$$

$$\hat{u}(\xi, t) = \hat{f}(\xi) e^{-\xi^2 t}$$

convolution!

$$u(x,t) = \mathcal{F}^{-1}\{\hat{f}(\xi)\} * \mathcal{F}^{-1}\{e^{-\xi^2 t}\}$$

$$= f(x) * \sqrt{\frac{a}{\pi}}\, e^{-ax^2}$$

reciprical since we're off by a constant

$$\mathcal{F}\{e^{-ax^2}\} = \sqrt{\frac{\pi}{a}}\, e^{-\xi^2/4a}$$

$$\frac{1}{4a} = t$$

$$\frac{1}{a} = 4t$$

$$a = \frac{1}{4t}$$

$$u(x,t) = f(x) * \sqrt{\frac{1/4t}{\pi}}\, e^{-x^2/4t}$$

$$\boxed{(f*g)(x) = \int_{-\infty}^{\infty} f(y)\, g(x-y)\, dy}$$

$$u(x,t) = f(x) * \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t}$$

Ans: $$\boxed{u(x,t) = \int_{-\infty}^{\infty} f(y) \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-y)^2}{4t}} dy}$$

Ex 4 $u_x + 3u_t = 0 \quad , -\infty < x < \infty \quad , t > 0$

$u(x,0) = f(x)$ ← assume f has a Fourier transform

$$\mathcal{F}\{u_x\} + 3\mathcal{F}\{u_t\} = 0$$

$$(-i\xi)\hat{u}(\xi,t) + 3\hat{u}_t(\xi,t) = 0$$

$$\hat{u}_t = \frac{i\xi}{3}\hat{u}$$

Separable!

$$\int \frac{d\hat{u}}{\hat{u}} = \int \frac{i\xi}{3} dt$$

$$\ln|\hat{u}| = \frac{i\xi}{3}t + C_1(\xi)$$

$$|\hat{u}| = e^{i\xi t/3 + C_1(\xi)}$$

$$\hat{u} = C_2(\xi) e^{i\xi t/3}$$

$$\mathcal{F}\{u(x,0) = f(x)\} \rightarrow \hat{u}(\xi,0) = \hat{f}(\xi)$$

$$\hat{u}(\xi,0) = C_2(\xi) e^{0} = \hat{f}(\xi)$$

$$\hat{u}(\xi,t) = \hat{f}(\xi) e^{i\xi t/3}$$

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$$\hat{u}(\xi,t) = \hat{f}(\xi)\, e^{i\xi t/3}$$

$$\mathcal{F}\{u(x+a)\} = e^{-ia\xi}\hat{u}(\xi)$$

$$-a = t/3$$

$$a = -t/3$$

$$u(x,t) = f(x+a)$$

Ans: $\boxed{u(x,t) = f(x - \tfrac{t}{3})}$

Ex 5 $u_t + \sin t\, u_x = 0, \quad -\infty < x < \infty, \; t > 0$

$u(x,0) = \sin x$

$$\mathcal{F}\{u_t\} + \mathcal{F}\{\sin t\, u_x\} = 0$$

$$\hat{u}_t(\xi,t) + \sin t\, \mathcal{F}\{u_x\} = 0$$

transform ignores t or y if we had $u(x,y)$ to start

$$\hat{u}_t + \sin t(-i\xi)\hat{u} = 0$$

$$\hat{u}_t = i\xi \sin t\, \hat{u} \qquad \text{Separable!}$$

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$$\int \frac{\partial \hat{u}}{\hat{u}} = \int i\xi \sin t\, dt$$

$$\ln|\hat{u}| = -i\xi\cos t + C_1(\xi)$$

$$\hat{u} = C_2(\xi)e^{-i\xi\cos t}$$

$$\mathcal{F}\{u(x,0)=\sin x\} \longrightarrow \hat{u}(\xi,0) = \underbrace{\mathcal{F}\{\sin x\}}$$

We don't have to find this! This is a little fun

$$\hat{u}(\xi,0) = C_2(\xi)e^{-i\xi\cos 0}$$

$$= C_2(\xi)e^{-i\xi} = \mathcal{F}\{\sin x\}$$

$$C_2(\xi) = \mathcal{F}\{\sin x\}e^{i\xi}$$

$$\hat{u}(\xi,t) = \mathcal{F}\{\sin x\}e^{i\xi}e^{-i\xi\cos t}$$

$$= \mathcal{F}\{\sin x\}e^{-i\xi(\cos t - 1)}$$

$$\mathcal{F}\{u(x+a)\} = e^{-ia\xi}\hat{u}(\xi)$$

same as

$$= e^{-ia\xi}\mathcal{F}\{u\}$$

$a = \cos t - 1$

$$u(x,t) = \sin(x+a)$$

Ans: $\boxed{u(x,t) = \sin(x + \cos t - 1)}$

$\boxed{\text{Ex 6}}$ $u_{tt} + 2u_t = -u \, , \; -\infty < x < \infty, \; t > 0$

$u(x,0) = f(x)$

$u_t(x,0) = g(x)$

$\mathcal{F}\{u_{tt}\} + 2\mathcal{F}\{u_t\} = -\mathcal{F}\{u\}$

$\hat{u}_{tt} + 2\hat{u}_t = -\hat{u}$

$\hat{u}_{tt} + 2\hat{u}_t + \hat{u} = 0$

$r^2 + 2r + 1 = 0$

$(r+1)^2 = 0$

$r = -1$

$\hat{u} = c_1(\xi) e^{-t} + c_2(\xi) t e^{-t}$

$ay'' + by' + cy = 0$
$ar^2 + br + c = 0$
If $b^2 - 4ac = 0$, case 2
with repeated roots
$r = -b/2a$
solution $y = c_1 e^{rt} + c_2 t e^{rt}$

$\mathcal{F}\{u(x,0) = f(x)\} \rightarrow \hat{u}(\xi, 0) = \hat{f}(\xi)$

$\hat{u}(\xi, 0) = c_1(\xi) e^0 + c_2(\xi) \cdot 0 \, e^0 = c_1(\xi) = \hat{f}(\xi)$

$\hat{u} = \hat{f}(\xi) e^{-t} + c_2(\xi) t e^{-t}$

Other condition is $u_t(x,0) = g(x)$ so

$\mathcal{F}\{u_t(x,0) = g(x)\} \rightarrow \hat{u}_t(\xi, 0) = \hat{g}(\xi)$

$\hat{u}_t = -\hat{f}(\xi) e^{-t} + c_2(\xi) \cdot 1 \, e^{-t} - c_2(\xi) t e^{-t}$

$\hat{u}_t(\xi, 0) = -\hat{f}(\xi) e^0 + c_2(\xi) e^0 - \cancel{c_2(\xi) \cdot 0 \, e^0} = \hat{g}(\xi)$

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$$C_2(\xi) = \hat{g}(\xi) + \hat{f}(\xi)$$

$$\hat{u}(\xi, t) = \hat{f}(\xi)e^{-t} + (\hat{g}(\xi) + \hat{f}(\xi))\,te^{-t}$$

ts don't change so...

Ans: $$\boxed{u(x,t) = f(x)e^{-t} + (g(x) + f(x))\,te^{-t}}$$

Examples mostly from Partial Differential Equations with Fourier series & BVP by Nakhlé Asmar