Laplace Transforms for solving PDEs

Technique:
1. Laplace of PDE
2. Solve the "ODE"
3. Inverse Laplace

Examples mostly from Partial Differential Equations with Fourier Series & Boundary Value Problems by Nakhlé Asmar

$\boxed{\text{Ex 1}}$

$u_t = u_{xx} \quad 0<x<\infty, \ t>0$

$u(0,t)=70, \ t>0$

$u(x,0)=0, \ 0<x,\infty$

Heat eqn

$\mathcal{L}\{u_t\} = \mathcal{L}\{u_{xx}\}$

Only t changes!

$sU(x,s) - \underbrace{\cancel{u(x,0)}}_{0} = U_{xx}(x,s)$

$sU = U_{xx}$

$U_{xx} - sU = 0$

2nd order homogeneous
$ay''+by'+cy=0$
$ar^2+br+c=0$

$r^2 - s = 0$

$r = \pm\sqrt{s}$

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Recall case 1: $b^2-4ac > 0$
solution to $ay'' + by' + cy = 0$ is
$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$ where $r = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$$U = \underbrace{C_1(s)}_{} e^{-\sqrt{s}\,x} + C_2(s) e^{\sqrt{s}\,x}$$

$U(x,s)$ so we don't just have a constant C_1 like we did in your previous d.e. course

as $x \to \infty$ we need U to still be bounded so $C_2(s)$ needs to be 0 so $e^{\sqrt{s}x}$ doesn't blow up.

$$U(x,s) = C_1(s) e^{-\sqrt{s}\,x}$$

We were also given $u(0,t) = 70$

$$\mathcal{L}\{u(0,t)\} = \mathcal{L}\{70\}$$

using our table:

$$U(0,s) = \frac{70}{s}$$

$$U(0,s) = C_1(s) e^{0} = C_1(s) = \frac{70}{s}$$

$$U(x,s) = \frac{70}{s} e^{-\sqrt{s}\,x}$$

Inverse Laplace using our table!

Ans: $\boxed{u(x,t) = 70\left(1 - \operatorname{erf}\left(\frac{x}{2\sqrt{t}}\right)\right)}$

Table:
$\mathcal{L}\{1 - \operatorname{erf}(\frac{a}{2\sqrt{t}})\} = \frac{e^{-a\sqrt{s}}}{s}$

our $a = x$ & 70 just tags along based on the linearity of the transform.

$\boxed{\text{Ex 2}}$ $u_{tt} = u_{xx} + t$, $x > 0$, $t > 0$

$u(0,t) = 0$, $t > 0$

$u(x,0) = 0$, $u_t(x,0) = 0$, $x > 0$

$$\mathcal{L}\{u_{tt}\} = \mathcal{L}\{u_{xx}\} + \mathcal{L}\{t\}$$

$$\underbrace{s^2 U(x,s) - s\cancel{u(x,0)}^{0} - \cancel{u_t(x,0)}^{0}} = U_{xx}(x,s) + \frac{1}{s^2}$$

Table lists: $\mathcal{L}\{u''(t)\} = s^2 U(s) - s u(0) - u'(0)$

$$s^2 U = U_{xx} + \frac{1}{s^2}$$

$$U_{xx} - s^2 U = -\frac{1}{s^2}$$

$$r^2 - s^2 = 0$$

$$r = \pm s$$

$$\underbrace{U_c}_{\text{complementary solution}} = C_1(s) e^{sx} + C_2(s) e^{-sx}$$

as $x \to \infty$ U_c needs to be bounded so $C_1(s) = 0$

$$U_c = C_2(s) e^{-sx}$$

2nd order nonhomogeneous

$ay'' + by' + cy = f(t)$

Solve with light undetermined coefficients

$ay'' + by' + cy = 0$

$ar^2 + br + c = 0$

y_c is the solution to:

$ay'' + by' + cy = 0$

y_p is anything that solves $ay'' + by' + cy = f(t)$

Ans: $y = y_c + y_p$

As far as x is concerned, s is a constant so $-\frac{1}{s^2}$ is a constant (w.r.t. x)

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$$U_p = A$$

$$\underbrace{(U_p)_x}_{\text{particular solution}} = 0$$

$$(U_p)_{xx} = 0$$

$$U_{xx} - s^2 U = -\frac{1}{s^2}$$

$$0 - s^2 A = -\frac{1}{s^2}$$

$$A = \frac{1}{s^4}$$

General solution to $U_{xx} - s^2 U = -\frac{1}{s^2}$ is

$$U = U_c + U_p$$

$$U = C_2(s) e^{-sx} + \frac{1}{s^4}$$

We were given $u(0,t) = 0$, $\mathcal{L}\{u(0,t)\} = \mathcal{L}\{0\}$

$$U(0,s) = 0$$

so $U(0,s) = C_2(s) e^0 + \frac{1}{s^4} = 0$

$$\longrightarrow C_2(s) = \frac{-1}{s^4}$$

$$U(x,s) = \frac{-1}{s^4} e^{-sx} + \frac{1}{s^4}$$

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$$U(x,s) = \frac{-1}{s^4} e^{-sx} + \frac{1}{s^4}$$

Tables: $\mathcal{L}\{\underbrace{H(t-a)}_{\text{Unit step (AKA Heaviside)}} u(t-a)\} = e^{-as} U(s)$

$$\mathcal{L}^{-1}\{e^{-sx}\underbrace{\left(\frac{-1}{s^4}\right)}_{U(s)}\} = H(t-x)\,u(t-x)$$

$$= H(t-x)\left[-\frac{(t-x)^3}{6}\right]$$

a

$$\mathcal{L}^{-1}\left\{\frac{-1}{s^4}\right\} = \frac{-1}{3!} t^3 = \frac{-t^3}{6}$$

If we have $e^{-as} U(s)$ find $\mathcal{L}^{-1}\{U(s)\} = u(t)$ then write it as $u(t-a)$ so for this problem $-\frac{(t-x)^3}{6}$

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

Ans: $$\boxed{u(x,t) = H(t-x)\left[-\frac{(t-x)^3}{6}\right] + \frac{t^3}{6}}$$

Ex 3 $u_{tt} = u_{xx} - g, \quad x>0, \; t>0$

$u(0,t) = 0, \quad t>0$

$u(x,0) = 0, \quad u_t(x,0) = 1, \quad x>0$

$$\mathcal{L}\{u_{tt}\} = \mathcal{L}\{u_{xx}\} - \mathcal{L}\{g\}$$

$$s^2 U(x,s) - s\,\cancel{u(x,0)}^{0} - \cancel{u_t(x,0)}^{1} = U_{xx}(x,s) - \frac{g}{s}$$

$$s^2 U - 1 = U_{xx} - \frac{g}{s}$$

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$$U_{xx} - s^2 U = \frac{g}{s} - 1$$

$$r^2 - s^2 = 0$$

$$r = \pm s$$

$$U_c = C_1(s) e^{sx} + C_2(s) e^{-sx}$$

$$C_1(s) = 0 \text{ so } e^{sx} \not\to \infty$$

$$U_c = C_2(s) e^{-sx}$$

$U_p = A$, since $\frac{g}{s} - 1$ is a constant as far as x is concerned

$$(U_p)_x = 0$$

$$(U_p)_{xx} = 0$$

$$0 - s^2 A = \frac{g}{s} - 1$$

$$A = -\frac{g}{s^3} + \frac{1}{s^2}$$

$$U(x,s) = \underbrace{C_2(s) e^{-sx}}_{U_c} \underbrace{- \frac{g}{s^3} + \frac{1}{s^2}}_{U_p}$$

$$\mathcal{L}\{u(0,t) = 0\} \longrightarrow U(0,s) = 0$$

$$U(0,s) = C_2(s) e^{0} - \frac{g}{s^3} + \frac{1}{s^2}$$

$$C_2(s) = -\left(-\frac{g}{s^3} + \frac{1}{s^2}\right) = \frac{g}{s^3} - \frac{1}{s^2}$$

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$$U(x,s) = \underbrace{\left(\frac{9}{s^3} - \frac{1}{s^2}\right)}_{F(s)} e^{-sx} - \frac{9}{s^3} + \frac{1}{s^2}$$

$$\mathcal{L}^{-1}\left\{\frac{9}{s^3} - \frac{1}{s^2}\right\} = \frac{9}{2}t^2 - t$$

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

Look at the s terms & adjust
$\frac{1}{s^3} \to t^2$ is involved
$\mathcal{L}\{t^2\} = \frac{2!}{s^{2+1}} = \frac{2}{s^3}$ we don't have the 2 so we divide

$$u(x,t) = H(t-x)F(t-x) - \frac{9}{2}t^2 + t$$

Ans: $\boxed{u(x,t) = H(t-x)\left[\frac{9}{2}(t-x)^2 - (t-x)\right] - \frac{9}{2}t^2 + t}$

Ex 4

$u_{tt} = u_{xx}$, $x > 0$, $t > 0$

$u(0,t) = \sin t$, $t > 0$

$u(x,0) = 0$, $u_t(x,0) = 1$, $x > 0$

Just realized we did this in class !!

$$\mathcal{L}\{u_{tt}\} = \mathcal{L}\{u_{xx}\}$$

$$s^2 U(x,s) - s\cancel{u(x,0)}^{0} - \cancel{u_t(x,0)}^{1} = U_{xx}(x,s)$$

$$s^2 U - 1 = U_{xx}$$

$$U_{xx} - s^2 U = -1$$

This is the same as the last problem so

$$U_c(x,s) = C_2(s)e^{-sx}$$

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$U_p = A$

so like before (I'm skipping steps since this is the 3rd problem w/ the same U_c & U_p. Don't skip steps on a test!!)

$-s^2 A = -1$

$A = \frac{1}{s^2}$

$U(x,s) = C_2(s)e^{-sx} + \frac{1}{s^2}$

$\mathcal{L}\{u(0,t) = \sin t\} = U(0,s) = \frac{1}{s^2+1}$

$U(0,s) = C_2(s)e^{0} + \frac{1}{s^2} = \frac{1}{s^2+1}$

$C_2(s) = \frac{1}{s^2+1} - \frac{1}{s^2}$

$U(x,s) = \left(\frac{1}{s^2+1} - \frac{1}{s^2}\right)e^{-sx} + \frac{1}{s^2}$

$\mathcal{L}^{-1}\left\{\frac{1}{s^2+1} - \frac{1}{s^2}\right\} = \sin t - t = f(t)$

$u(x,t) = H(t-x)\ f(t-x) + t$

where $F(s) = \frac{1}{s^2+1} - \frac{1}{s^2}$

Ans: $\boxed{u(x,t) = H(t-x)\left[\sin(t-x) - (t-x)\right] + t}$

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$\boxed{\text{Ex 5}}$ $u_t = u_{xx}$, $0 < x < \infty$, $t > 0$

$u(0,t) = 100\,H(t-2)$, $t > 0$

$u(x,0) = 0$, $0 < x < \infty$

$$\mathcal{L}\{u_t\} = \mathcal{L}\{u_{xx}\}$$

$$sU(x,s) - \overset{0}{\cancel{u(x,0)}} = U_{xx}(x,s)$$

$$U_{xx} - sU = 0$$

$$r^2 - s = 0$$

$$r = \pm\sqrt{s}$$

$$U(x,s) = C_1(s)e^{\sqrt{s}\,x} + C_2(s)e^{-\sqrt{s}\,x}$$

$C_1(s) = 0$ to keep U bounded

$$U(x,s) = C_2(s)e^{-\sqrt{s}\,x}$$

$$\mathcal{L}\{u(0,t) = 100\,H(t-2)\}$$

$$U(0,s) = \frac{100e^{-2s}}{s}$$

$$U(0,s) = C_2(s)e^{0} = \frac{100e^{-2s}}{s}$$

$$U(x,s) = \frac{100e^{-2s}}{s}e^{-\sqrt{s}\,x} = \underbrace{\frac{100e^{-\sqrt{s}\,x}}{s}}_{F(s)}\;\underbrace{e^{-2s}}_{e^{-as}}$$

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$$\mathcal{L}^{-1}\left\{\frac{100e^{-\sqrt{s}x}}{s}\right\} = \underbrace{\left(1-\text{erf}\left(\frac{x}{2\sqrt{t}}\right)\right)100}_{f(t)}$$

$$u(x,t) = H(t-2)\,f(t-2)$$

Ans: $$\boxed{u(x,t) = H(t-2)\left[1-\text{erf}\left(\frac{x}{2\sqrt{t-2}}\right)\right]100}$$

Recall $H(t-2) = \begin{cases} 0 & t<2 \\ 1 & t\geq 2 \end{cases}$

so we could also write this as

$$u(x,t) = 0 \quad , \quad 0<t<2$$
$$u(x,t) = 100\left(1-\text{erf}\left(\frac{x}{2\sqrt{t-2}}\right)\right) \quad , \quad t\geq 2$$

Ex 6 $\quad u_t = u_{xx} \quad x>0, t>0$

$u(0,t) = a \quad , \quad t>0$

$u(x,0) = b$

$$\mathcal{L}\{u_t\} = \mathcal{L}\{u_{xx}\}$$

$$sU(x,s) - u(x,0) = U_{xx}(x,s)$$

$$U_{xx} - sU = -b$$

$$r^2 - s = 0$$
$$r = \pm\sqrt{s}$$

$$U_c = C_1(s)e^{\sqrt{s}x} + C_2(s)e^{-\sqrt{s}x}$$

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$C_1(s) = 0$

$U_c = C_2(s) e^{-\sqrt{s}\,x}$

$U_p = A$

$(U_p)_x = 0$

$(U_p)_{xx} = 0$

$-As = -b$

$A = \frac{b}{s}$

$U(x,s) = C_2(s) e^{-\sqrt{s}\,x} + \frac{b}{s}$

$\mathcal{L}\{u(0,t) = a\} = U(0,s) = \frac{a}{s} = C_2(s) e^{0} + \frac{b}{s}$

$C_2(s) = \frac{a}{s} - \frac{b}{s} = \frac{(a-b)}{s}$

$U(x,s) = \frac{(a-b)}{s} e^{-\sqrt{s}\,x} + \frac{b}{s}$

Ans: $\boxed{u(x,t) = (a-b)\left(1 - \operatorname{erf}\left(\frac{x}{2\sqrt{t}}\right)\right) + b}$