Fun with Fourier

In class we found the formula for the Fourier series of $f(x)$ on the interval $-L<x<L$ to be

$$\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)+b_n \sin\left(\frac{n\pi x}{L}\right), \quad -L<x<L$$

where the Fourier coefficients are

$$a_n=\frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \quad n=0,1,2,\ldots$$

$$b_n=\frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx, \quad n=1,2,3,\ldots$$

Recall if $f(x)$ is even ($f(-x)=f(x)$, ex constants, $x^2, x^4, \ldots$, $\cos x$) $b_n=0$ & if $f(x)$ is odd ($f(-x)=-f(x)$, ex $x, x^3, x^5, \ldots, \sin x$) $a_n=0$.

$\boxed{\text{Ex 1}}$ Find the Fourier series of $f(x)=x, -1<x<1$

L

$f(x)$ is odd so $a_n=0$

$$b_n=\underbrace{\frac{1}{1}}_{L}\int_{-1}^{1} \underbrace{\underbrace{x}_{\text{odd}}\,\underbrace{\sin\left(\frac{n\pi x}{1}\right)}_{\text{odd}}}_{\text{even}}dx = 2\int_0^1 x\sin(nx\pi)\,dx$$

Integration by parts or tabular integration by parts if you prefer

LIATE

$u=x$

sign	u & its derivatives	dV & its antiderivatives
+	x	$\sin n\pi x$
−	1	$-\frac{1}{n\pi}\cos n\pi x$
+	0	$-\frac{1}{n^2\pi^2}\sin n\pi x$

$dV=\sin(n\pi x)\,dx$

$$=2\left[x\left(\frac{-1}{n\pi}\cos n\pi x\right)-1\left(\frac{-1}{n^2\pi^2}\sin n\pi x\right)\right]_0^1$$

$$=2\left[\frac{-1}{n\pi}\cos n\pi+\cancel{\frac{1}{n^2\pi^2}\sin n\pi}-\left[\cancel{0\left(\frac{-1}{n\pi}\cos 0\right)}+\cancel{\frac{1}{n^2\pi^2}\sin}\right.\right.$$

$$b_n = 2\left[\frac{-1}{n\pi}\cos n\pi\right] = \frac{2}{n\pi}(-1)^{n+1}$$

$n=1 \quad \cos n\pi \rightarrow \cos\pi = -1$

$n=2 \quad \cos 2\pi = 1$

the $+1$ is to include the negative from $\frac{-1}{n\pi}$

Fourier coeff: $a_n = 0$

$$b_n = \frac{2}{n\pi}(-1)^{n+1}$$

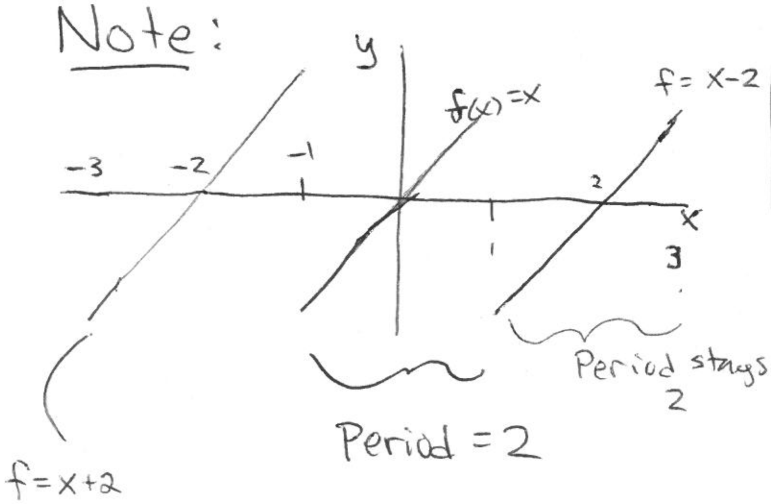
Fourier series: $\sum_{n=1}^{\infty} \underbrace{\left(\frac{2}{n\pi}(-1)^{n+1}\right)}_{b_n} \underbrace{\sin\left(\frac{n\pi x}{1}\right)}_{\sin\frac{n\pi x}{L}}$, $\quad -1<x<1$

If they wanted us to write out the first few terms:

$$\frac{2}{\pi}\left[\underbrace{\frac{1}{1}\sin\pi x}_{n=1} - \underbrace{\frac{1}{2}\sin(2\pi x)}_{n=2} + \underbrace{\frac{1}{3}\sin(3\pi x)}_{n=3} - \cdots\right], \quad -1<x<1$$

pulling this out from the $\frac{2}{n\pi}$

Note:



We can extend $f(x)$ on $-L<x<L$ by just repeating our graph
Called a periodic extension

Ex 2 Find the Fourier Series of $f(x) = |\sin \pi x|$

Note: $f(-x) = |\sin(-\pi x)| = |\underbrace{-\sin(\pi x)}_{\text{sine is an odd function}}| = |\sin \pi x| = f(x)$

Although sine is odd, $|\sin \pi x|$ is even

This means $b_n = 0$

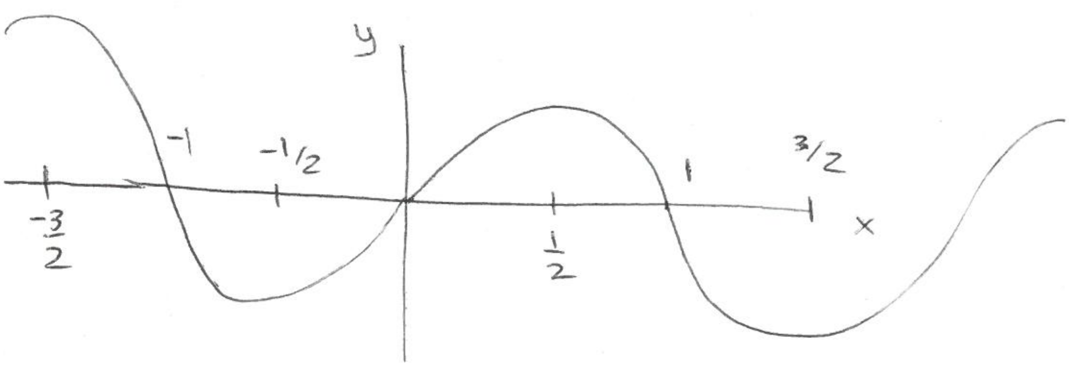
$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \quad n = 0, 1, 2, \ldots$$

$f(x) = |\sin \pi x|$

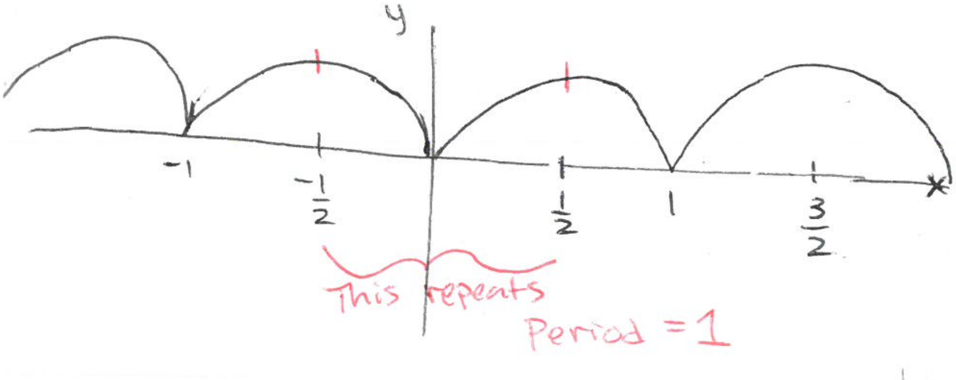
Note: $x = 0$, $f(0) = |\sin 0| = 0$

$x = \frac{1}{2}$, $f(\frac{1}{2}) = |\sin \frac{\pi}{2}| = 1$

$x = \frac{3}{2}$, $f(\frac{3}{2}) = |\sin \frac{3\pi}{2}| = 1$



$f = \sin \pi x$



← Pretend these are the same height

$f = |\sin \pi x|$

41

$$a_n = \frac{1}{\underbrace{1/2}_{L}} \int_{-1/2}^{1/2} |\sin \pi x| \cos\left(\frac{n\pi x}{1/2}\right) dx$$

$$= 2 \int_{-1/2}^{1/2} \underbrace{\underbrace{|\sin \pi x|}_{\text{even}} \underbrace{\cos(2n\pi x)}_{\text{even}}}_{\text{even}} dx$$

$$= 2 \cdot 2 \int_0^{1/2} \underbrace{|\sin \pi x|} \cos(2n\pi x) dx$$

$\sin \pi x \geq 0$ on $0 < x < \frac{1}{2}$

$$= 4 \int_0^{1/2} \sin \pi x \cos 2n\pi x \, dx$$

Product to sum

$$\boxed{\sin a \cos b = \frac{1}{2}\left(\sin(a+b) + \sin(a-b)\right)}$$

$$= 4 \int_0^{1/2} \frac{1}{2}\left[\sin(\pi x + 2n\pi x) + \sin(\pi x - 2n\pi x)\right] dx$$

$$a_n = 4 \int_0^{1/2} \frac{1}{2}\left[\sin(\pi x(2n+1)) + \sin(\pi x(1-2n))\right] dx$$

$$a_0 = \underbrace{4 \cdot \frac{1}{2}} \int_0^{1/2} \left[\sin(\pi x(0+1)) + \sin(\pi x(1-0)) \, dx\right.$$

$$= 2 \int_0^{1/2} 2 \sin \pi x \, dx$$

$$= \frac{-4}{\pi} \cos \pi x \Big|_0^{1/2} = -\frac{4}{\pi}\left[\overset{0}{\cancel{\cos \frac{\pi}{2}}} - \overset{1}{\cancel{\cos 0}}\right] = \frac{4}{\pi}$$

$$n \neq 0: \; a_n = 2\left[-\frac{1}{(2n+1)\pi} \cos(\pi x \underbrace{(2n+1)}_{\text{always odd}}) - \frac{1}{(1-2n)\pi} \cos(\pi x \underbrace{(1-2n)}_{\text{always odd}})\right]_0^{1/2}$$

$$= 2\left[\cancel{\frac{-1}{(2n+1)\pi} \overset{0}{\cos}\left(\frac{\pi}{2}(2n+1)\right)} - \cancel{\frac{1}{(1-2n)\pi} \overset{0}{\cos}\left(\frac{\pi}{2}(1-2n)\right)} - \left(\frac{-1}{(2n+1)\pi} \cos 0 - \frac{1}{(1-2n)\pi} \cos 0\right)\right]$$

5

$$a_n = 2\left[\frac{1}{(2n+1)\pi} + \frac{1}{(1-2n)\pi}\right] = \frac{2}{\pi}\left[\frac{1-2n+2n+1}{(2n+1)(1-2n)}\right] = \frac{2}{\pi}\left[\frac{2}{1-4n^2}\right]$$

$$= \underbrace{\frac{4}{\pi}\left[\frac{1}{1-4n^2}\right]}$$

Not sure if this is actually nicer than what we started with

Fourier series: $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$ $L = 1/2$

$$\boxed{\frac{2}{\pi} + \sum_{n=1}^{\infty}\left(\frac{4}{\pi}\left(\frac{1}{1-4n^2}\right)\right)\cos(2n\pi x)}$$

<u>Definition</u>: Let $f(x)$ be given on $0<x<L$

The <u>odd extension</u> of f is $f_o(x) = \begin{cases} f(x), & 0<x<L \\ -f(-x), & -L<x<0 \end{cases}$

letter o

The <u>even extension</u> of f is $f_e(x) = \begin{cases} f(x), & 0<x<L \\ f(-x), & -L<x<0 \end{cases}$

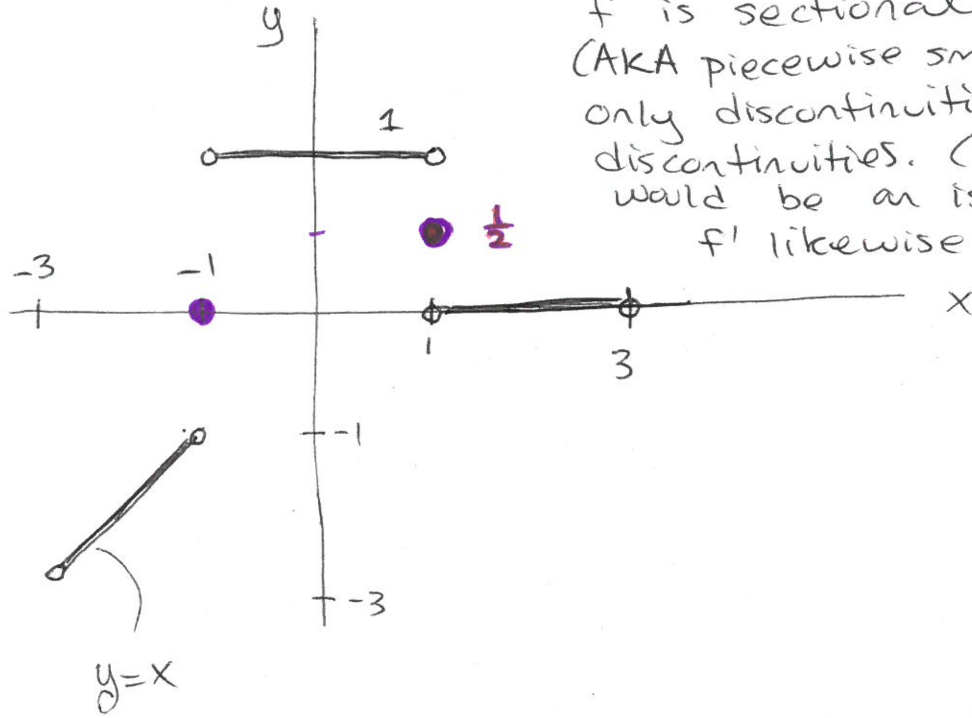
<u>Theorem</u>: If $f(x)$ & $f'(x)$ are piecewise continuous then the Fourier series of $f(x)$ converges to $\frac{1}{2}[f(x^+) + f(x^-)]$

average value from the right & left

Ex 3

For $f(x) = \begin{cases} 0, & 1 < x < 3 \\ 1, & -1 < x < 1 \\ x, & -3 < x < -1 \end{cases}$

& determine if it is sectionally smooth. If so, find what its Fourier series converges to at each point x in the interval & at the endpoints. Sketch.



f is sectionally smooth (AKA piecewise smooth) since its only discontinuities are jump discontinuities. (Vertical asymptotes would be an issue. Holes are fine.) f' likewise is fine.

If the Fourier series for f(x) is C(x):

$C(-1) = \frac{1}{2}(-1 + 1) = 0$

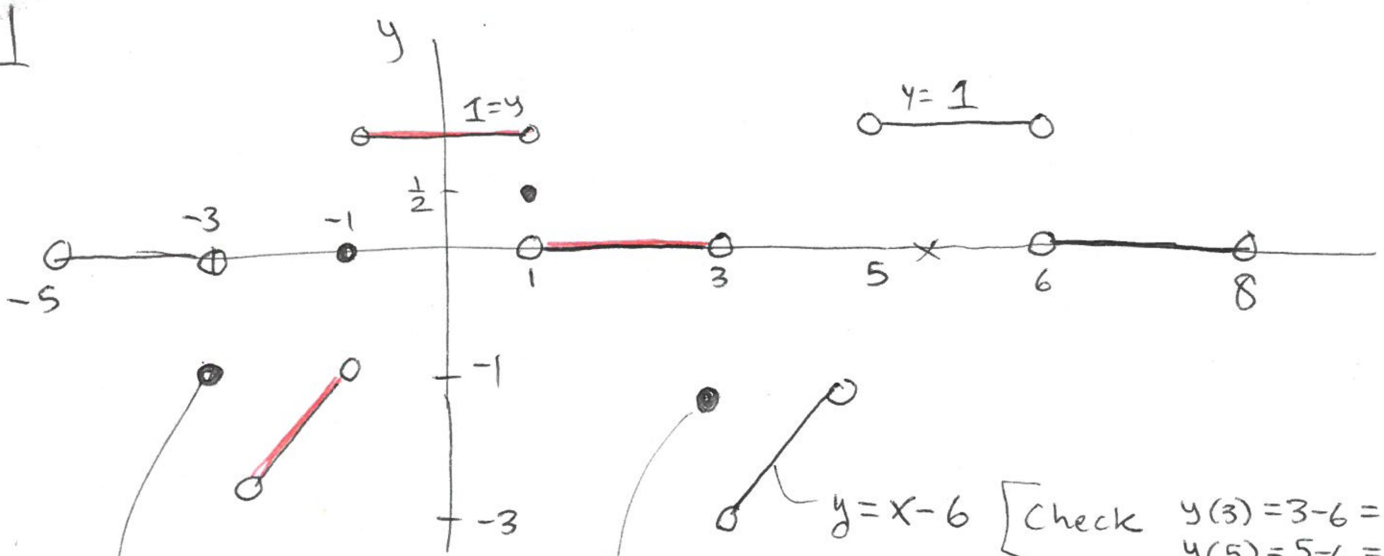
$y=1$ at $x=-1$

$y=x$ at $x=-1$

$C(1) = \frac{1}{2}(1+0) = \frac{1}{2}$

At $x=-3$ & $x=3$ we need to do a periodic expansion of f, so repeating f starting at $x=3$ on the right

7|



$y = x-6$ [Check $y(3) = 3-6 = -3$ ✓
$y(5) = 5-6 = -1$ ✓
Alternatively, $f(x) = x$ goes to $f(x-2L)$]

$$C(3) = \frac{1}{2}\left(0 + (-3)\right) = -\frac{3}{2}$$

$$C(-3) = \frac{1}{2}\left(0 + (-3)\right) = -\frac{3}{2}$$

our endpoints were $x = \pm 3$, $x = \pm 1$
on inbetween spots it goes to the value of $f(x)$

Recall from class:

<u>Fourier sine series</u> for any integrable function $f(x)$ on $[0, L]$ is

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right), \quad b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx, \quad n=1,2,\ldots$$

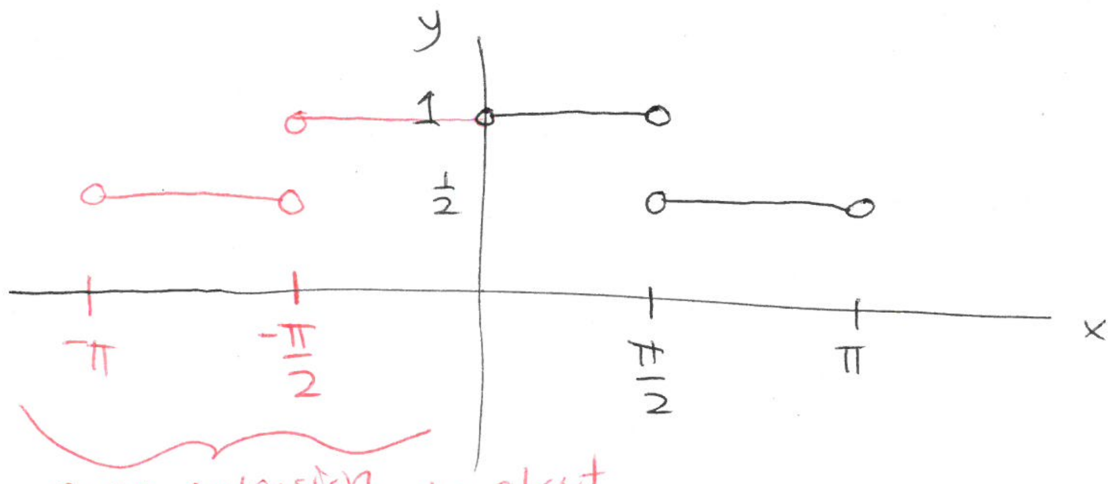
<u>Fourier cosine series</u> for f on $[0, L]$ is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}, \quad a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx,$$

$n = 0, 1, 2, 3$

$\boxed{\text{Ex 4}}$ Sketch the even perioditic extension of $f(x) = \begin{cases} 1, & 0 < x < \frac{\pi}{2} \\ \frac{1}{2}, & \frac{\pi}{2} < x < \pi \end{cases}$

Find its Fourier cosine series. To what values does the series converge at $x=0$, $x = \frac{\pi}{2}$, $x = \pi$, $x = 3\pi/2$, & $x = 2\pi$?



Note: even has symmetry about the y-axis

$$f_e(x) = \begin{cases} f(x), & 0 < x < L \\ f(-x), & -L < x < 0 \end{cases}$$

$$f(-x) = \begin{cases} 1, & 0 < -x < \frac{\pi}{2} \\ \frac{1}{2}, & \frac{\pi}{2} < -x < \pi \end{cases} \longrightarrow \begin{cases} 1 & -\frac{\pi}{2} < x < 0 \\ \frac{1}{2} & -\pi < x < -\frac{\pi}{2} \end{cases}$$

$$a_n = \frac{2}{\underset{\uparrow L}{\pi}} \int_0^{\pi} f(x) \cos \frac{n\cancel{\pi}x}{\cancel{\pi}}\, dx = \frac{2}{\pi}\left[\int_0^{\pi/2} 1 \cos nx\, dx + \int_{\pi/2}^{\pi} \frac{1}{2} \cos nx\, dx\right]$$

If $n=0$: $a_0 = \frac{2}{\pi}\left[\int_0^{\pi/2} \cancel{\cos 0}^{\,1}\, dx + \int_{\pi/2}^{\pi} \frac{1}{2} \cancel{\cos 0}^{\,1}\, dx\right]$

$$= \frac{2}{\pi}\left[\pi/2 + \frac{1}{2}\left[\pi - \pi/2\right]\right] = \frac{2}{\pi}\left[\frac{3\pi}{4}\right] = \frac{3}{2}$$

a) $n \neq 0$ $a_n = \frac{2}{\pi}\left[\int_0^{\pi/2} \cos nx\,dx + \int_{\pi/2}^{\pi} \frac{1}{2}\cos nx\,dx\right]$

$= \frac{2}{\pi}\left[\frac{1}{n}\sin nx\Big|_0^{\pi/2} + \frac{1}{2n}\sin nx\Big|_{\pi/2}^{\pi}\right]$

$= \frac{2}{\pi}\left[\frac{1}{n}\sin\frac{n\pi}{2} - \cancel{\frac{1}{n}\sin 0} + \cancel{\frac{1}{2n}\sin n\pi} - \frac{1}{2n}\sin\frac{n\pi}{2}\right]$

$= \frac{\cancel{2}}{\pi}\left[\underbrace{\left(\frac{1}{n}-\frac{1}{2n}\right)}_{\frac{1}{\cancel{2}n}}\sin\frac{\pi n}{2}\right]$

$= \frac{1}{n\pi}\sin\frac{n\pi}{2}$

$n=1$, $\sin\frac{\pi}{2} = 1$
$n=2$, $\sin\pi = 0$
$n=3$, $\sin\frac{3\pi}{2} = -1$
$n=4$, $\sin 2\pi = 0$

even terms are 0

$= \frac{1}{(2n-1)\pi}(-1)^{n+1}$

tried $2n+1$ for $n=1$ that gives 3 so I adjusted.

Fourier cosine series:

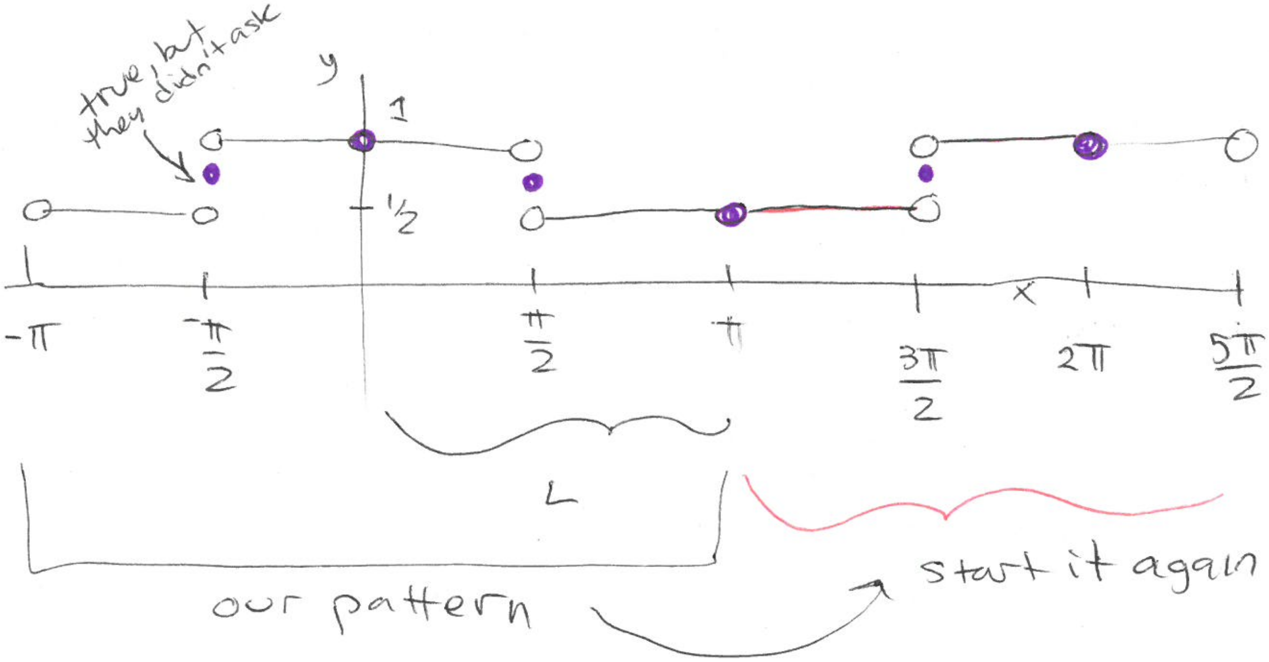
$$\underbrace{\frac{3}{4}}_{\frac{a_0}{2}} + \sum_{n=1}^{\infty} \frac{1}{(2n-1)\pi}(-1)^{n+1}\cos((2n-1)x)$$

or as terms written out:

$\cos\left(\frac{n\pi x}{L}\right) = \cos nx \longrightarrow \cos((2n-1)x)$

$$\frac{3}{4} + \underbrace{\frac{1}{\pi}\cos x}_{n=1} - \underbrace{\frac{1}{3\pi}\cos 3x}_{n=2} + \underbrace{\frac{1}{5\pi}\cos 5x}_{n=3} - \cdots$$

10



If $C(x)$ is our Fourier cosine series

$$C(0) = \frac{1}{2}\left[f(\underset{0}{x^-}) + f(\underset{0}{x^+})\right] = \frac{1}{2}\left[1+1\right] = 1$$

$$C\left(\frac{\pi}{2}\right) = \frac{1}{2}\left[f\left(\frac{\pi}{2}^-\right) + f\left(\frac{\pi}{2}^+\right)\right] = \frac{1}{2}\left[1+\frac{1}{2}\right] = \frac{3}{4}$$

$$C(\pi) = \frac{1}{2}\left[f(\pi-) + f(\pi+)\right] = \frac{1}{2}\left[\frac{1}{2}+\frac{1}{2}\right] = \frac{1}{2}$$

$$C\left(\frac{3\pi}{2}\right) = \frac{1}{2}\left[f\left(\frac{3\pi}{2}-\right) + f\left(\frac{3\pi}{2}+\right)\right] = \frac{1}{2}\left[\frac{1}{2}+1\right] = \frac{3}{4}$$

$$C(2\pi) = \frac{1}{2}\left[f(2\pi-) + f(2\pi+)\right] = \frac{1}{2}\left[1+1\right] = 1$$

$\boxed{\text{Ex 5}}$ Find the Fourier sine series of the function defined by $f(x) = 1-2x,\ 0<x<1$. Sketch the graph of the odd periodic extension of $f(x)$ & determine the sum of the series where the graph has a jump.

$b_n = \frac{2}{1}\int_0^1 (1-2x)\sin\frac{n\pi x}{1}\,dx$ Tabular or reg integration by parts

Sign	u & its derivatives	dV & its antiderivatives
+	$1-2x$	$\sin n\pi x$
−	-2	$-\frac{1}{n\pi}\cos n\pi x$
+	0	$-\frac{1}{n^2\pi^2}\sin n\pi x$

$$b_n = 2\left[(1-2x)\left(-\frac{1}{n\pi}\cos n\pi x\right) - (-2)\left(-\frac{1}{n^2\pi^2}\sin n\pi x\right)\right]_0^1$$

$$= 2\left[(-1)\left(-\frac{1}{n\pi}\cos n\pi\right) - \cancel{\frac{2}{n^2\pi^2}\overset{0}{\sin n\pi}} - \left[(1-\cancel{2\cdot 0})\left(-\frac{1}{n\pi}\right)\overset{1}{\cancel{\cos 0}}\right]\right]$$

$$= 2\left[\frac{1}{n\pi}\cos n\pi + \frac{1}{n\pi}\right] = \frac{2}{n\pi}\underbrace{\left[\cos n\pi + 1\right]}$$

$n=1$: $-1+1=0$
$n=2$: $1+1=2$
$n=3$: $-1+1=0$

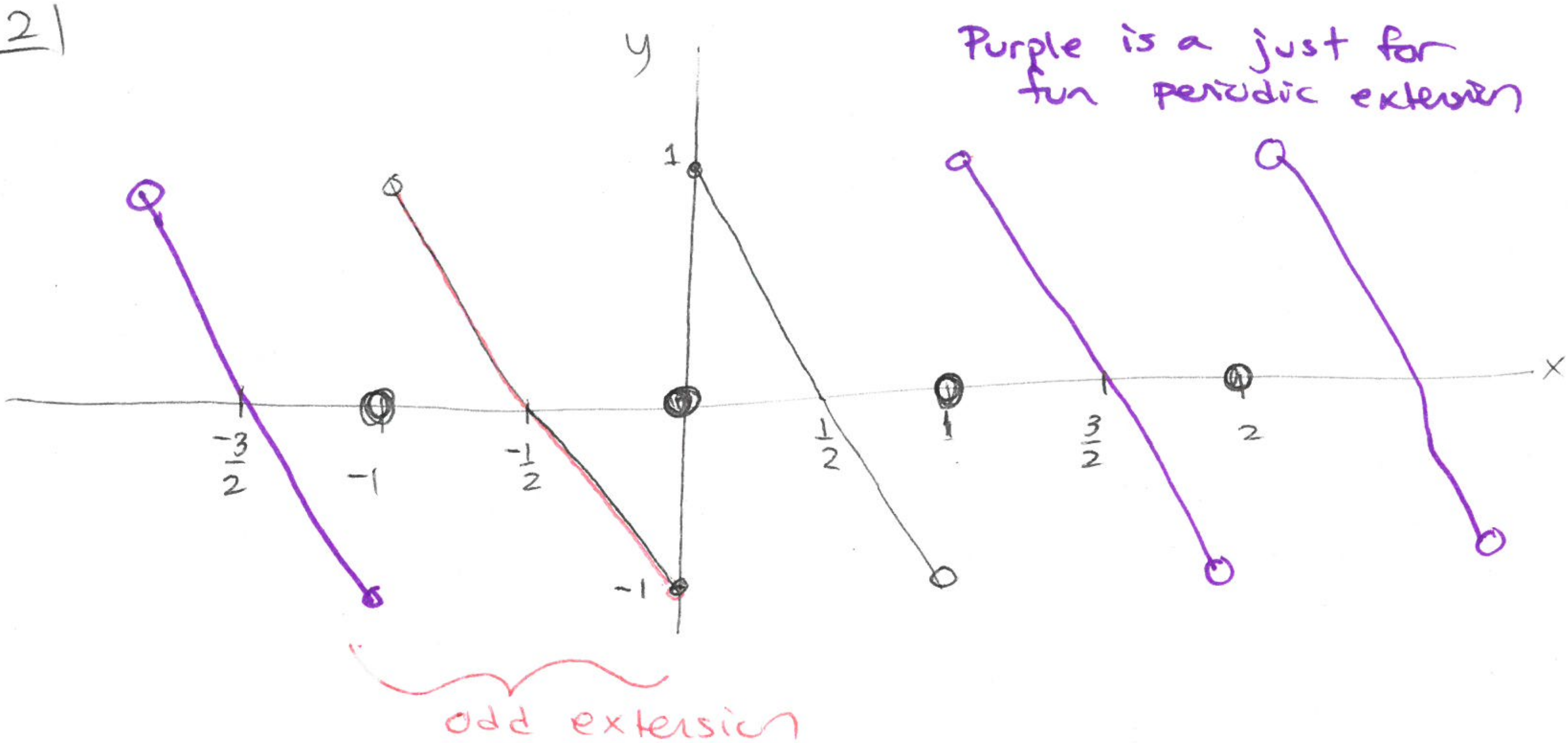
$$= \frac{\cancel{2}}{\underset{n}{\cancel{2n}}\pi}[2] = \frac{2}{n\pi}$$

2n since we want even terms

Fourier sine series: $\displaystyle\sum_{n=1}^{\infty}\frac{2}{n\pi}\sin\left(\frac{2n\pi x}{1}\right)$

☆ If they didn't care about us simplifying!

$$\sum_{n=1}^{\infty}\frac{2}{n\pi}\left[\cos n\pi + 1\right]\sin\frac{n\pi x}{1}$$



Odd extension of $f = f_o(x) = \begin{cases} f(x) & 0<x<L \\ -f(-x), & -L<x<0 \end{cases}$

$f(x) = 1-2x \qquad -f(-x) = -(1-2(-x))$
$\qquad\qquad\qquad\qquad\qquad = -1-2x$ } Note: $x=0, y=-1$; $x=-1/2, y=0$

Sum of the series $C(x) = \frac{1}{2}\left[f(x^-)+f(x^+)\right]$

$C(0) = \frac{1}{2}\left[f(0^-)+f(0^+)\right] = \frac{1}{2}\left[-1+1\right] = 0$

For this problem C at every jump should be 0.

☆ Find the sum of the Fourier sine serie at $x=-.7$ (Just added a question to make it more interesting)

$C(-.7) = \frac{1}{2}\left[f(-.7^-)+f(-.7^+)\right] = -1-2(-.7) = .4$

Examples mostly from Boundary Value Problems by David Powers