MA 401 Test 2 No Work=No Credit!

1. (15 points) Use d'Alembert's formula to solve the Cauchy wave equation:

$u_{tt} = 4u_{xx}, \qquad x \in \Re,\ t > 0$

$u(x,0) = 7e^{-x^2}$

$u_t(x,0) = x^2$

2. (17 points) Solve the following Cauchy heat equation:

$u_t - u_{xx} = 0, \quad x \in \Re,\ t > 0$

$$u(x,0) = \begin{cases} 6, & |x| < 3 \\ 0, & |x| \geq 3 \end{cases}$$

Write your answer in terms of the error function.

3. (23 points) Find the bounded solution u(x,t) using Laplace Transforms

$u_{tt} = 4u_{xx} + 1, \quad x > 0,\ t > 0$

$u(0,t) = 3t^2$

$u(x,0) = 0$

$u_t(x,0) = 0$

4. (23 points) Find the solution u(x,t) using Fourier Transforms

$u_t = \cos(t)\, u_x, \quad x \in \Re,\ t > 0$

$u(x,0) = \frac{1}{2}e^{-|x|}$

5. (22 points) Find the first 4 nonzero terms in a power series expansion about $x = 0$ for the solution to the Initial Value Problem: $y'' + 2xy' + 2y = 0;\ y(0) = 1,\ y'(0) = 0$

Your work should include a recurrence relation.

Table of Fourier Transforms

$u(x)$	$\mathcal{F}[u(x)] = \hat{u}(\xi) = \int_{-\infty}^{\infty} u(x) e^{i\xi x} dx$
$\frac{1}{2} e^{-\lvert x \rvert}$	$\frac{1}{1+\xi^2}$
e^{-ax^2}	$\sqrt{\frac{\pi}{a}}\, e^{-\left(\frac{\xi^2}{4a}\right)}$
xe^{-ax^2}	$\frac{1}{2a}(i\xi)\sqrt{\frac{\pi}{a}}\, e^{-\left(\frac{\xi^2}{4a}\right)}$
$\left(\frac{a}{\pi}\right)\frac{1}{x^2+a^2}$	$e^{-a\lvert \xi \rvert}$
$u(x+a)$	$e^{-ia\xi}\hat{u}(\xi)$
$u'(x)$	$(-i\xi)\hat{u}(\xi)$
$u''(x)$	$(-i\xi)^2\ \hat{u}(\xi)$
$(u * v)$	$\hat{u}(\xi)\hat{v}(\xi)$

Table of Laplace Transforms

$u(t)$	$\mathcal{L}[u(t)] = U(s)$
1	$\frac{1}{s}$
$e^{at}u(t)$	$U(s-a)$
$\mathcal{H}(t-a)$	$\frac{e^{-as}}{s}$
$u(t-a)\mathcal{H}(t-a)$	$e^{-as}U(s)$
t^n	$\frac{n!}{s^{n+1}}$
$\sin at$	$\frac{k}{s^2+a^2}$
$\cos at$	$\frac{s}{s^2+a^2}$
e^{at}	$\frac{1}{s-a}$
$\sinh at$	$\frac{a}{s^2-a^2}$
$\cosh at$	$\frac{s}{s^2-a^2}$
$\frac{1}{\sqrt{t}}e^{-a^2/4t}$	$\sqrt{\frac{\pi}{s}}e^{-a\sqrt{s}}$
$\frac{a}{\sqrt{4t^3}}e^{-a^2/4t}$	$\sqrt{\pi}e^{-a\sqrt{s}}$
$\operatorname{erf}\left(\sqrt{t}\right)$	$\frac{1}{s\sqrt{1+s}}$
$1-\operatorname{erf}\left(\frac{a}{2\sqrt{t}}\right)$	$\frac{e^{-a\sqrt{s}}}{s}$
$u'(t)$	$sU(s)-u(0)$
$u''(t)$	$s^2U(s)-su(0)-u'(0)$

MA 401 T2 Solutions

1. (15 points)

$$u(x,t)=\frac{1}{2}\left[7e^{-(x-2t)^2}+7e^{-(x+2t)^2}\right]$$

$$+\frac{1}{4}\int_{x-2t}^{x+2t} s^2\,ds$$

$$=\frac{1}{2}\left[7e^{-(x-2t)^2}+7e^{-(x+2t)^2}\right]+\frac{1}{12}\left[(x+2t)^3-(x-2t)^3\right]$$

2. (15 points)

$$u(x,t)=\int_{-3}^{3} 6\,\frac{1}{\sqrt{4\pi t}}\,e^{-\frac{(x-y)^2}{4t}}\,dy$$

$$r=\frac{x-y}{2\sqrt{t}} \qquad dr=-\frac{1}{2\sqrt{t}}\,dy$$

$$6\int_{\frac{x+3}{2\sqrt{t}}}^{\frac{x-3}{2\sqrt{t}}} -\frac{1}{\sqrt{\pi}}\,e^{-r^2}\,dr = 6\int_{\frac{x-3}{2\sqrt{t}}}^{\frac{x+3}{2\sqrt{t}}} \frac{1}{\sqrt{\pi}}\,e^{-r^2}\,dr$$

$$=\frac{6}{2}\left[\operatorname{erf}\left(\frac{x+3}{2\sqrt{t}}\right)-\operatorname{erf}\left(\frac{x-3}{2\sqrt{t}}\right)\right]$$

3. (20 points)

$$s^2 U(x,s) - \cancel{s u(x,0) - u_t(x,0)} = 4U_{xx} + \frac{1}{s}$$

$$4U_{xx} - s^2 U = -\frac{1}{s}$$

$$4r^2 - s^2 = 0$$

$$r = \pm \frac{s}{2}$$

$$U_c = C_1(s) e^{-\frac{s}{2}x} + \cancel{C_2(s) e^{\frac{s}{2}x}}$$

$$U_p = A \quad (U_p)_x = 0 \quad (U_p)_{xx} = 0$$

$$-s^2 A = -\frac{1}{s} \quad A = \frac{1}{s^3}$$

$$U = C_1(s) e^{-\frac{s}{2}x} + \frac{1}{s^3}$$

$$\mathcal{L}\{u(0,t) = 3t^2\} \rightarrow U(0,s) = \frac{6}{s^3}$$

$$U(0,s) = C_1(s) + \frac{1}{s^3} = \frac{6}{s^3}$$

$$C_1(s) = \frac{5}{s^3}$$

$$U = \underbrace{\frac{5}{s^3}}_{F(s)} \underbrace{e^{-\frac{s}{2}x}}_{e^{-as}} + \frac{1}{s^3}$$

$$a = \frac{x}{2}$$

$$f(t) = \frac{5}{2}t^2$$

$$u(x,t) = H\left(t - \frac{x}{2}\right) \frac{5}{2}\left(t - \frac{x}{2}\right)^2 + \frac{1}{2}t^2$$

4. (20 points)

$$\mathcal{F}\{u_t\} = \mathcal{F}\{\cos t \, u_x\}$$

$$\hat{u}_t = \cos t \, (-i\xi)\,\hat{u}$$

$$\int \frac{\partial \hat{u}}{\hat{u}} = \int -i\xi \cos t \, dt$$

$$\ln|\hat{u}| = -i\xi \sin t + c_1(\xi)$$

$$\hat{u} = c_2(\xi)\, e^{-i\xi \sin t}$$

$$\mathcal{F}\{u(x,0) = \tfrac{1}{2} e^{-|x|}\} \rightarrow \hat{u}(\xi, 0) = \frac{1}{1+\xi^2}$$

$$\hat{u}(\xi, 0) = c_2(\xi) e^{0}$$

$$\hat{u} = \frac{1}{1+\xi^2} e^{-i\xi \underbrace{\sin t}_{a}}$$

$$\boxed{u(x,t) = \tfrac{1}{2} e^{-|x+\sin t|}}$$

5. (20 points)

$$y = \sum_{n=0}^{\infty} c_n x^n \qquad y' = \sum_{n=0}^{\infty} c_n n x^{n-1} \qquad y'' = \sum_{n=0}^{\infty} c_n n(n-1) x^{n-2}$$

$$\sum_{n=0}^{\infty} c_n n(n-1) x^{n-2} + \sum_{n=0}^{\infty} 2 c_n n x^n + \sum_{n=0}^{\infty} 2 c_n x^n = 0$$

$k = n-2$

$n = k+2$

$$\sum_{k=0}^{\infty} c_{k+2}(k+2)(k+1) x^k + \sum_{k=0}^{\infty} (2c_k)(k+1) x^k = 0$$

$$c_{k+2} = \frac{-2c_k \cancel{(k+1)}}{(k+2)\cancel{(k+1)}}$$

$y(0) = 1 \rightarrow c_0 = 1$

$y'(0) = 0 \rightarrow c_1 = 0$

$k=0$: $c_2 = \frac{-2c_0}{2} = -1$

$k=1$: $c_3 = \frac{-2c_1}{3} = 0$

$k=2$: $c_4 = \frac{-2c_2}{4} = \frac{1}{2}$

$k=3$: $c_5 = \frac{-2c_3}{5} = 0$

$k=4$: $c_6 = \frac{-2c_4}{6} = \frac{-1}{3} \cdot \frac{1}{2} = -\frac{1}{6}$

$$\boxed{1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6}$$