Sturm-Liouville Problems

Regular SLP

Given an ODE

$$\frac{d}{dx}\left[r(x)\frac{dy}{dx}\right] + q(x)y + \lambda p(x)y = 0, \quad a<x<b$$

and boundary conditions

$$a_1 y(a) + a_2 y'(a) = 0$$

$$b_1 y(b) + b_2 y'(b) = 0$$

r, r', q, p are continuous on $[a,b]$

a_1 & a_2 aren't both 0

b_1 & b_2 aren't both 0

$p(x)$ & $r(x) > 0$ on $[a,b]$

Nontrivial solutions $y(x)$ are called eigenfunctions & the number λ that corresponds with that solution is an eigenvalue

* FYI, the ODE above can be written differently. Don't get attached to the letters of the various functions. -- just focus on the general form of the ODE

$\boxed{\text{Ex 1}}$ Find all eigenvalues & eigen functions of
$-y'' = \lambda y \quad y(0) = y(2) = 0$

* Some books really like writing the ODE in the form $-y'' = \_\_\_$

$-y'' = \lambda y$

$y'' + \lambda y = 0$ Note for our general SLP: $r(x) = 1,\ q(x) = 0,\ p(x) = 1$

* We will have a 2nd order ODE for these problems. A common 1st step is finding the characteristic equation

$r^2 + \lambda = 0$

$r^2 = -\lambda$

$r = \pm\sqrt{-\lambda}$

* Depending on the value of λ we will get different cases.

Recall $ay'' + by' + cy = 0 \rightarrow ar^2 + br + c = 0$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$b^2 - 4ac > 0$ real roots r_1 & r_2:
$y = C_1 e^{r_1 x} + C_2 e^{r_2 x}$

$b^2 - 4ac = 0$ repeated roots r:
$y = C_1 e^{rx} + C_2 x e^{rx}$

$b^2 - 4ac < 0$ complex roots $r = \alpha \pm \beta i$
$y = e^{\alpha x}[C_1 \cos\beta x + C_2 \sin\beta x]$

3

$y'' + \lambda y = 0 \qquad y(0) = y(2) = 0$

$r = \pm\sqrt{-\lambda}$

* If $\lambda = 0 \rightarrow r = \pm\sqrt{-0} = 0$, repeated roots

$y = c_1 e^{0x} + c_2 x e^{0x}$

$y = c_1 + c_2 x$

* Plug in boundary conditions

$y(0) = c_1 + c_2 \cdot 0 = c_1 = 0$

$y(2) = c_1 + c_2 \cdot 2 = 0$ but $c_1 = 0 \rightarrow 2c_2 = 0$
$\rightarrow c_2 = 0$

so $y = 0 + 0x = 0$, trivial solution

we won't include $\lambda = 0$ as one of our eigenvalues

* If $\lambda < 0$, say $\lambda = -\alpha^2 \rightarrow r = \pm\sqrt{-(-\alpha^2)} = \pm\alpha$,
distinct roots

$y = c_1 e^{\alpha x} + c_2 e^{-\alpha x}$

* Plug in boundary conditions

$y(0) = c_1 e^0 + c_2 e^0 = c_1 + c_2 = 0 \rightarrow c_1 = -c_2$

$y(2) = c_1 e^{2\alpha} + c_2 e^{-2\alpha} = 0$

$-c_2 e^{2\alpha} + c_2 e^{-2\alpha} = 0$

$c_2 (e^{-2\alpha} - e^{2\alpha}) = 0$

either $c_2 = 0$ or $e^{-2\alpha} - e^{2\alpha} = 0$

$c_2 = 0 \rightarrow c_1 = -c_2 = 0 \rightarrow y = 0e^{\alpha x} + 0e^{-\alpha x} = 0$ trivial sol

but $e^{-2\alpha} - e^{2\alpha} = 0 \rightarrow e^{-2\alpha} = e^{2\alpha} \rightarrow \alpha = 0 \rightarrow \lambda = -0^2$ but $\lambda < 0$

So λ is not < 0, since again we got the trivial solution

* FYI, frequently we switch to hyperbolic trig functions in the case $\lambda = -\alpha^2$ $r = \pm\alpha$ like we did in class to get $y = c_1 \cosh \alpha x + c_2 \sinh \alpha x$

$y(0) = c_1 \cosh 0 + c_2 \sinh 0 = c_1 \cdot 1 = 0$

$y(2) = \underbrace{\cancel{c_1 \cosh 2\alpha}}_{c_1 = 0} \underbrace{+ c_2 \sinh 2\alpha = 0}$

so $c_2 = 0 \rightarrow y = 0$ or

$\sinh 2\alpha = 0 \rightarrow \alpha = 0 \rightarrow \lambda = 0$

but $\lambda < 0$

For this problem with our fairly simple boundary conditions it didn't really matter if we used $y = c_1 e^{\alpha x} + c_2 e^{-\alpha x}$ or $y = c_1 \cosh \alpha x + c_2 \sinh \alpha x$ but for more complicated conditions it will

* Last case, $\lambda > 0$ say $\lambda = \alpha^2 \rightarrow r = \pm\sqrt{-\alpha^2} = \pm \alpha i$

(If having α in the β position of our formula is confusing either do $\lambda = \beta^2 \rightarrow r = \pm\sqrt{-\beta^2} = \pm\beta i$ or think of r in the complex case as $r = a \pm bi$)

$y = e^{ax}[c_1 \cos bx + c_2 \sin bx]$)

$y = e^{0x}[c_1 \cos \alpha x + c_2 \sin \alpha x]$

$y = c_1 \cos \alpha x + c_2 \sin \alpha x$

* Plug in boundary conditions

$y(0) = c_1 \cos 0 + c_2 \sin 0 = c_1 = 0 \rightarrow y = c_2 \sin \alpha x$

$y(2) = c_2 \sin 2\alpha = 0$

5

$c_2 \sin 2\alpha = 0$ either $c_2 = 0$ which gives us the trivial solution $y=0$ ✗

or $\sin 2\alpha = 0$

* Some $\underbrace{\text{minor (??)}}_{\text{You decide if this is only minor trickiness!}}$ trickiness: when we said $\lambda = \alpha^2$ we assumed $\alpha > 0$ since we wanted $\lambda > 0$. We could have stated that. So $\alpha > 0$ but we know $\sin y = 0$ for $y = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$

$\alpha > 0$ means we are only interested in $y = \pi, 2\pi, 3\pi, \ldots$

since we have $\sin 2\alpha = 0$ to include $\sin \pi$, $\sin 3\pi$, $\sin 5\pi$, etc we need $\alpha = \frac{n\pi}{2}$ to get $\sin 2\left(\frac{n\pi}{2}\right) = \sin n\pi$

$n = 1, 2, 3, \ldots$

$\alpha_n = \frac{n\pi}{2}$, $n = 1, 2, 3, \ldots$

helps us to figure out if we want $\alpha_7 = \frac{7\pi}{2}$, etc

$\lambda = \alpha^2$ so $\boxed{\lambda_n = \left(\frac{n\pi}{2}\right)^2 \Big\} \text{ eigenvalue} \quad n = 1, 2, 3, \ldots}$

$y = c_1 \cos \alpha x + c_2 \sin \alpha x$ but we found $c_1 = 0$

so $\boxed{y_n(x) = c_2 \sin\left(\left(\frac{n\pi}{2}\right)x\right) \Big\} \text{ eigenfunction} \quad n = 1, 2, 3, \ldots}$

Sometimes people do C_n here

$\boxed{\text{Ex 2}}$ Find all the real eigenvalues & eigenfunctions

$y'' + \lambda y = 0 \qquad y(-3) = y(3) \qquad y'(-3) = y'(3)$

$r^2 + \lambda = 0$

$r = \pm\sqrt{-\lambda}$

* All we need to do is run through the cases $\lambda > 0$, $\lambda = 0$, $\lambda < 0$. The order we do this is up to you.

$\lambda = 0$: $r^2 = \pm\sqrt{-0} = 0 \rightarrow y = c_1 + c_2 x$

$y(-3) = c_1 + c_2(-3) \quad = \quad y(3) = c_1 + 3c_2$

$c_2(-3) = 3c_2 \rightarrow c_2 = 0 \rightarrow y = c_1 + 0x$
$y = c_1$

$y' = 0 \quad$ so $\quad y'(-3) = 0 \quad y'(3) = 0$

$\underbrace{\lambda_0 = 0}_{\text{eigenvalue}}, \quad \underbrace{y_0(x) = c_1}_{\text{eigenfunction}}$, where c_1 is just a constant.

* $\lambda < 0$, say $\lambda = -\alpha^2$ $(\alpha > 0)$ $\quad r = \pm\sqrt{-(-\alpha^2)}$
$r = \pm\alpha$

$y = c_1\cosh(\alpha x) + c_2\sinh(\alpha x)$

$y(-3) = c_1\cosh(-3\alpha) + c_2\sinh(-3\alpha) = c_1\cosh(3\alpha) - c_2\sinh(3\alpha)$

$y(3) = c_1\cosh(3\alpha) + c_2\sinh(\alpha 3)$

FYI: $\cosh(-x) = \dfrac{e^{(-x)} + e^{-(-x)}}{2} = \cosh x$ & $\sinh(-x) = \dfrac{e^{(-x)} - e^{-(-x)}}{2} = \dfrac{e^{-x} - e^{x}}{2}$
$= -\sinh(x)$

7

$y(-3) = y(3)$

$\cancel{c_1\cosh(3\alpha)} - c_2\sinh(3\alpha) = \cancel{c_1\cosh(3\alpha)} + c_2\sinh(3\alpha)$

$0 = 2c_2\sinh(3\alpha)$ so $c_2 = 0$ or $\sinh(3\alpha) = 0 \to \alpha = 0$ $\to \lambda = 0$ $\lambda < 0$ $\to c_2 = 0$

$$\left.\begin{aligned} y' &= c_1\alpha\sinh(\alpha x) + c_2\alpha\cosh(\alpha x) \\ y'(-3) &= c_1\alpha\sinh(-3\alpha) + c_2\alpha\cosh(-3\alpha) \\ &= -c_1\alpha\sinh(3\alpha) + c_2\alpha\cosh(3\alpha) \\ y'(3) &= c_1\alpha\sinh(3\alpha) + c_2\alpha\cosh(3\alpha) \end{aligned}\right\}$$

$c_2 = 0$, I just didn't plug it in!

$y'(-3) = y'(3)$

$-c_1\alpha\sinh(3\alpha) + \cancel{c_2\alpha\cosh(3\alpha)} = c_1\alpha\sinh(3\alpha) + \cancel{c_2\alpha\cosh 3\alpha}$

$0 = 2c_1\alpha\sinh(3\alpha)$

$c_1 = 0$

since $\alpha > 0$

$y = 0$ trivial solution!

* Last case $\lambda > 0$, $\lambda = \alpha^2$ $(\alpha > 0)$ $r = \pm\sqrt{-(\alpha^2)}$
$= \pm\alpha i$

$y = c_1\cos\alpha x + c_2\sin\alpha x$

$y(-3) = c_1\underbrace{\cos(-3\alpha)}_{} + c_2\underbrace{\sin(-3\alpha)}_{} = c_1\cos 3\alpha - c_2\sin 3\alpha$

cos is an even function $\cos(-x) = \cos x$

sin is an odd function $\sin(-x) = -\sin x$

Neat that we saw on the last page that cosh is even & sinh is odd.

8

$$y(3) = C_1 \cos 3\alpha + C_2 \sin 3\alpha$$

$$y(-3) = y(3)$$

$$\cancel{C_1 \cos 3\alpha} - C_2 \sin 3\alpha = \cancel{C_1 \cos 3\alpha} + C_2 \sin 3\alpha$$

$$2C_2 \sin 3\alpha = 0 \rightarrow C_2 = 0 \text{ or } 3\alpha = n\pi,\ n=1,2,3,\ldots$$

$$y' = -\alpha \sin \alpha x\, C_1 + \alpha \cos \alpha x\, C_2$$

I forgot to write these ü

$$y'(-3) = -\alpha C_1 \sin(-3\alpha) + C_2 \alpha \cos(-3\alpha)$$
$$= C_1 \alpha \sin(3\alpha) + C_2 \alpha \cos(3\alpha)$$

$$y'(3) = -\alpha \sin 3\alpha\, C_1 + \alpha \cos 3\alpha\, C_2$$

$$y'(-3) = y'(3)$$

$$C_1 \alpha \sin(3\alpha) + \cancel{C_2 \alpha \cos 3\alpha} = -\alpha \sin 3\alpha\, C_1 + \cancel{\alpha \cos 3\alpha\, C_2}$$

$$C_1 2\alpha \sin 3\alpha = 0 \rightarrow C_1 = 0 \text{ or } 3\alpha = n\pi,\ n=1,2,\ldots$$
$(\alpha > 0 \text{ so } \alpha \neq 0)$

We want nontrivial solutions so

$$3\alpha = n\pi \longrightarrow \alpha = \frac{n\pi}{3},\ n = 1,2,3,\ldots$$

$\lambda = \alpha^2$ so $\lambda_n = \left(\frac{n\pi}{3}\right)^2$ } eigenvalues $n=1,2,3,\ldots$

$y_n(x) = C_1 \cos\left(\frac{n\pi}{3}x\right) + C_2 \sin\left(\frac{n\pi}{3}x\right)$ } eigenfunction

If we want to be fancy, we also had $\lambda_0 = 0$ $y_0(x) = C_1$ if we plug $n=0$ into our above eigenvalues & eigenfunctions note that $\lambda_n = \left(\frac{n\pi}{3}\right)^2$, $y_n(x) = C_1 \cos\left(\frac{n\pi}{3}x\right) + C_2 \sin\left(\frac{n\pi}{3}x\right)$, $n = 0,1,2,3,\ldots$ covers it all. Also, instead of C_1 & C_2 we could do b_n & C_n to emphasize that the values of the constants can be different

9

$\boxed{\text{Ex } 3}$ $y'' + 3y + \lambda y = 0 \quad y'(0) = 0 \quad y'(\pi) = 0$

* Be careful here. This doesn't say
$y'' + 3y' + \lambda y = 0$

$$y'' + (3+\lambda)y = 0$$
$$r^2 + (3+\lambda) = 0$$
$$r^2 = -(3+\lambda)$$
$$r = \pm\sqrt{-(3+\lambda)}$$

* $3+\lambda < 0 \rightarrow 3+\lambda = -\alpha^2, \ \alpha > 0$
$\rightarrow r = \pm\sqrt{-(-\alpha^2)} = \pm\alpha$

$y = c_1 \cosh \alpha x + c_2 \sinh(\alpha x)$

$y'(x) = c_1 \alpha \sinh \alpha x + c_2 \alpha \cosh \alpha x$

$y'(0) = \cancel{c_1 \alpha \sinh 0} + c_2 \alpha \cosh 0 = 0$

$c_2 \alpha = 0 \quad \alpha > 0 \rightarrow c_2 = 0$

$y'(\pi) = c_1 \alpha \sinh \alpha \pi + \cancel{c_2 \alpha \cosh \alpha \pi} = 0$

$c_1 \alpha \sinh \alpha \pi = 0$

$\sinh x = \dfrac{e^x - e^{-x}}{2} = 0 \rightarrow x = 0$

For $\sinh \alpha\pi = 0 \rightarrow \alpha = 0$ but $\alpha > 0$

$\rightarrow c_1 = 0$

$c_1 = c_2 = 0 \rightarrow$ trivial solution

* $3+\lambda=0 \rightarrow r=\pm\sqrt{-(0)}=0$

$y=c_1+c_2x$

$y'=c_2 \qquad y'(0)=c_2=0 \rightarrow c_2=0 \rightarrow y=c_1$

Note: $y'(\pi)=0$ doesn't give us more info for this case

$3+\lambda=0 \rightarrow \underbrace{\lambda_0=-3}_{\text{eigen value}} \qquad \underbrace{y_0=c_1}_{\text{eigenfunction}}$

* $3+\lambda>0 \rightarrow 3+\lambda=\alpha^2, \quad \alpha>0$

$r=\pm\sqrt{-\alpha^2}=\pm\alpha i$

$y=c_1\cos\alpha x+c_2\sin\alpha x$

$y'=-c_1\alpha\sin\alpha x+c_2\alpha\cos\alpha x$

$y'(0)=0=\cancel{-c_1\alpha\sin 0}+c_2\alpha\cos 0=c_2\alpha \qquad \alpha>0$

$\rightarrow c_2=0$

$y=c_1\cos\alpha x$

$y'=-c_1\alpha\sin\alpha x$

$y'(\pi)=-c_1\alpha\sin\alpha\pi=0 \qquad$ if $c_1=0 \rightarrow$ trivial solution

$\alpha>0$ so we need

$\sin\alpha\pi=0$

so $\alpha\pi=n\pi \quad n=1,2,3,\ldots$

$\alpha=n$

$3+\lambda=\alpha^2 \rightarrow 3+\lambda=n^2$

$\underbrace{\lambda_n=n^2-3}_{\text{eigenvalue}}$, & $\underbrace{y_n=c_1\cos nx}_{\text{eigenfunction}}$, $n=1,2,\ldots$

* Like the last example:

We have $\lambda_0 = -3 \quad y_0 = c_1$

$\& \ \lambda_n = n^2 - 3 \quad y_n = c_1 \cos nx \quad \} \ n = 1, 2, \ldots$

If we plug $n=0$ into λ_n & y_n we get

$\lambda_0 = 0^2 - 3 = -3 \quad y_0 = c_1 \cos 0 = c_1$

so we can cover both variations with

$\lambda_n = n^2 - 3, \quad y_n = c_1 \cos nx$ with $n = 0, 1, 2, \ldots$

we could do c_n here if we wanted to stress that the constant for the $n=3$ case isn't necessarily the same for the $n=57$ case, etc

Ex 4 $y'' + y' - \lambda y = 0 \quad y(0) = 0 \quad y(1) = 0$

* This is not exactly our regular SLP

We could have had $\frac{d}{dx}\left[e^x y'\right] - \lambda e^x y = 0$

$\rightarrow e^x y' + e^x y'' - \lambda e^x y = 0 \rightarrow y'' + y' - \lambda y = 0$

but then our $p(x)$ is still < 0; however we are still looking at different values of λ.

Let's find our eigenvalues & eigenfunctions all the same.

12

$y'' + y' - \lambda y = 0 \quad y(0) = 0 \quad y(1) = 0$

$r^2 + r - \lambda = 0 \rightarrow r = \frac{-1 \pm \sqrt{1^2 - 4(1)(-\lambda)}}{2 \cdot 1}$

$r = \frac{-1 \pm \sqrt{1 + 4\lambda}}{2}$

* $1 + 4\lambda > 0$, let $1 + 4\lambda = \alpha^2$, $\alpha > 0$

$r = \frac{-1 \pm \sqrt{\alpha^2}}{2} = \frac{-1 \pm \alpha}{2}$ distinct real roots

$r_1 = \frac{-1 + \alpha}{2}$, $r_2 = \frac{-1 - \alpha}{2}$

$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$

$y(0) = c_1 e^0 + c_2 e^0 = c_1 + c_2 = 0 \rightarrow c_1 = -c_2$

$y(1) = c_1 e^{r_1} + c_2 e^{r_2} = 0$

$-c_2 e^{r_1} + c_2 e^{r_2} = 0$

$c_2 (e^{r_1} - e^{r_2}) = 0 \quad e^{r_1} - e^{r_2} \neq 0$

since $r_1 \neq r_2$

$\rightarrow c_2 = 0 \quad c_1 = -c_2$

$\rightarrow c_2 = 0$

$y = 0$ trivial solution

13

* $1+4\lambda=0 \rightarrow r=\dfrac{-1\pm\sqrt{0}}{2}=-\dfrac{1}{2}$

$y=c_1e^{-1/2 x}+c_2xe^{-1/2 x}$

$y(0)=c_1e^0+c_2\cdot 0e^0=c_1=0$

$y=c_2xe^{-1/2 x}$

$y(1)=0=c_2\cdot 1e^{-\frac{1}{2}\cdot 1} \rightarrow c_2=0$

trivial solution

* $1+4\lambda<0$, let $1+4\lambda=-\alpha^2$, $\alpha>0$

$r=\dfrac{-1\pm\sqrt{-\alpha^2}}{2}=\underbrace{-\dfrac{1}{2}}_{a}\pm\underbrace{\dfrac{\alpha}{2}}_{b}i$

$y=e^{ax}\left[c_1\cos bx+c_2\sin bx\right]$

$y(0)=e^0\left[c_1\cancel{\cos 0}+\cancel{c_2\sin 0}\right]=c_1=0$

$y=e^{ax}\left[c_2\sin bx\right]$

$y(1)=\underbrace{e^a}_{\text{never } 0}\left[\underbrace{c_2}_{}\sin b\right]=0$

if $c_2=0 \rightarrow$ trivial solution

so $\sin b=0 \rightarrow b=n\pi$ $n=1,2,\ldots$

$b=n\pi$ ($n\neq 0$ since $\alpha>0$ & $b=\alpha/2$)

$\dfrac{\alpha}{2}=n\pi \rightarrow \alpha_n=2n\pi$ $n=1,2,\ldots$

$$\alpha_n = 2n\pi \qquad n = 1, 2, 3, \ldots$$

$$1 + 4\lambda = -\alpha^2$$

$$1 + 4\lambda_n = -(2n\pi)^2 = -4n^2\pi^2$$

eigenvalues $\left\{ \lambda_n = \dfrac{-4n^2\pi^2 - 1}{4} \qquad n = 1, 2, 3, \ldots \right.$

eigenfunction $\left\{ y_n = e^{-1/2x}\left[C_2 \sin n\pi x\right] \qquad n = 1, 2, 3, \ldots \right.$

Ex 1 from *Applied Partial Differential Equations* by DuChateau & Zachmann

Ex 2&3 from *Fundamentals of Differential Equations & Boundary Value Problems* by Nagle, Saff, & Snider

Ex 4 from *Introduction to Applied Partial Differential Equations* by Davis

---

For practice & to fill up this last bit of space:

Ex 1 boundary conditions: $y(0) = y(2) = 0$ } Dirichlet since it is y on the boundary

Ex 2 boundary conditions: $y(-3) = y(3)$, $y'(-3) = y'(3)$ } Periodic

Ex 3 boundary conditions: $y'(0) = 0 \quad y'(\pi) = 0$ } Neumann since y' on boundary

Ex 4 boundary conditions: $y(0) = 0, \ y(1) = 0$ } Dirichlet