MA 401 Test 1 No Work=No Credit!

1. (25 points)

a) Solve the following Cauchy problem to find u(x,t):

$$\frac{t}{2}u_x + xu_t = xu + xe^t, \qquad x \in \Re,\ x > 0$$

$$u(x,0) = \frac{1}{x}, \qquad x \in \Re,\ x > 0$$

b) Is the PDE in part a) linear or nonlinear? If applicable, is it homogeneous or nonhomogeneous?

2. (15 points) Find the general solution u(x,t) in terms of two arbitrary functions to the PDE given by $u_{xt} - 3u_x = 0$

3. (15 points) Use $-5u_{xx} + 4u_{xt} + u_{tt} = 0$ to answer the following:

a) Classify this PDE as hyperbolic, parabolic, or elliptic

b) Find its general solution in terms of two arbitrary functions by making a change of variables

4. (30 points) Use $4u_{xx} - 4u_{xt} + u_{tt} + u_t = 0$ to answer the following:

a) Classify this PDE as hyperbolic, parabolic, or elliptic

b) Make a change of variables and write this PDE in its canonical form (Do not solve it). Be sure to show *all* of your work and clearly label your partial derivatives.

5. (15 points) In 2D, suppose $u = u(r,\theta)$ satisfies Laplace's equation $\Delta u = 0$ in the disk $0 \le r < 3$ and on the boundary satisfies $u(3,\theta) = e^{2\theta}$ where $0 \le \theta \le 2\pi$

a) What is the value of u at the origin?

MA 401 T1 Solutions

1. (25 points)

a) $\frac{t}{2x} u_x + u_t = u + e^t$

$$\frac{dx}{dt} = \frac{t}{2x}$$

$$\int 2x\,dx = \int t\,dt$$

$$x^2 = \frac{1}{2}t^2 + C$$

$$\xi = x^2 - \frac{1}{2}t^2, \quad \tau = t$$

$$u_\tau = u + e^\tau$$

$$u_\tau - u = e^\tau$$

$$\mu = e^{\int -1\,d\tau} = e^{-\tau}$$

$$\frac{\partial}{\partial \tau}\left[e^{-\tau} u\right] = 1$$

$$e^{-\tau} u = \tau + g(\xi)$$

$$u = e^\tau \tau + e^\tau g(\xi)$$

$$u(x,t) = e^t t + e^t g\left(x^2 - \frac{t^2}{2}\right)$$

$$u(x,0) = \frac{1}{x} = g(x^2) \rightarrow g(x) = \frac{1}{\sqrt{x}}$$

$$\boxed{u(x,t) = te^t + e^t\left(\frac{1}{\sqrt{x^2 - t^2/2}}\right)}$$

b) linear, nonhomogeneous

2. (15 points)

$$V = u_x$$

$$V_t - 3V = 0$$

$$V_t = 3V$$

$$\int \frac{\partial v}{v} = \int 3\, \partial t$$

$$\ln|v| = 3t + g(x)$$

$$|v| = e^{3t + g(x)}$$

$$v = H(x) e^{3t}$$

$$u_x = H(x) e^{3t}$$

$$\boxed{u(x,t) = u = h(x) e^{3t} + f(t)}$$

3. (15 points)

a) $D = B^2 - 4AC = 16 - 4(-5) \cdot 1 = 36 > 0$

hyperbolic

b) $\xi = x + \left(\frac{-4+6}{2}\right) t = x + t$

$\tau = x - 5t$ $\boxed{u(x,t) = g(x+t) + h(x-5t)}$

$u_{\xi\tau} = 0$

$u_\xi = G(\xi)$

$u = g(\xi) + h(\tau)$

4. (30 points)

a) $D = B^2 - 4AC = 16 - 4 \cdot 4 \cdot 1 = 0$ parabolic

b) $\xi = x \quad \tau = x - \frac{B}{2C} t = x + \frac{4}{2} t = x + 2t$

$u_x = u_\xi \xi_x + u_\tau \tau_x$

$= u_\xi \cdot 1 + u_\tau \cdot 1$

$u_{xx} = [u_\xi + u_\tau]_\xi \cdot 1 + [u_\xi + u_\tau]_\tau \cdot 1$

$= u_{\xi\xi} + u_{\tau\xi} + u_{\xi\tau} + u_{\tau\tau}$

$u_t = u_\xi \xi_t + u_\tau \tau_t$

$= u_\xi \cdot 0 + u_\tau (2)$

$= 2u_\tau$

$u_{tt} = [2u_\tau]_\tau \cdot 2 = 4u_{\tau\tau}$

$u_{xt} = [u_\xi + u_\tau]_\tau \cdot 2 = 2u_{\xi\tau} + 2u_{\tau\tau}$

$4u_{xx} - 4u_{xt} + u_{tt} + u_t = 0$

$4u_{\xi\xi} + \cancel{8u_{\xi\tau}} + 4u_{\tau\tau} - \cancel{8u_{\xi\tau}} - 8u_{\tau\tau} + \cancel{4u_{\tau\tau}} + 2u_\tau = 0$

$\boxed{4u_{\xi\xi} + 2u_\tau = 0}$

$\boxed{u_{\xi\xi} + \frac{1}{2}u_\tau = 0}$

5. (15 points)

a) $u(0,0) = \frac{1}{2\pi}\int_0^{2\pi} e^{2\theta}\, d\theta$

$= \frac{1}{2\pi} \cdot \frac{1}{2} e^{2\theta} \Big|_0^{2\pi}$

$= \frac{1}{4\pi}\left[e^{4\pi} - 1\right]$