MA 401 Test 3 No Work=No Credit!

1. (47 points) Use Separation of Variables to solve the following. Be sure to show all of your work and test all the cases.

$u_t = 3u_{xx}, \quad 0 < x < 1, t > 0$

$u_x(0,t) = u_x(1,t) = 0, t > 0$

$u(x,0) = 6 + 5\cos(2\pi x) - 7\cos(6\pi x)$

2. (38 points)

a) Find the Fourier sine series of $f(x) = \begin{cases} 2 - x, & 0 < x < 2 \\ 0, & 2 < x < 3 \end{cases}$

b) Graph the odd extension of f

c) What value does the series converge to when $x = -1.5$? When $x = 0$? When $x = 5$?

3. (15 points) Find the eigenvalues and eigenfunctions associated with $2 - \lambda < 0$

$y'' + (\lambda - 2)y = 0, \; y(\pi) = 0, \; y'(0) = 0$

MA 401 T3 solutions

1. (47 points)

$u = y(x)\,g(t)$

$u_t = 3u_{xx}$

$y g' = 3 y'' g$

$$\frac{g'}{3g} = \frac{y''}{y} = -\lambda$$

$y'' + \lambda y = 0$ $\quad u_x = y'(x) g(t)$

$u_x(0,t) = y'(0) g(t) = 0 \rightarrow y'(0) = 0$

$u_x(1,t) = y'(1) g(t) = 0 \rightarrow y'(1) = 0$

$r^2 + \lambda = 0$

$r = \pm\sqrt{-\lambda}$

$\lambda < 0,\ \lambda = -\alpha^2,\ r = \pm\alpha$

$y = c_1 \cosh \alpha x + c_2 \sinh \alpha x$

$y' = c_1 \alpha \sinh \alpha x + c_2 \alpha \cosh \alpha x$

$y'(0) = c_2 \alpha = 0 \rightarrow c_2 = 0$

$y'(1) = c_1 \alpha \sinh \alpha = 0 \rightarrow c_1 = 0$ trivial sol

$\lambda = 0,\ r = 0,$

$y = c_1 + c_2 x$

$y' = c_2 \quad y'(0) = c_2 = 0 \rightarrow c_2 = 0$

$\lambda_0 = 0,\ y_0 = c_0$

$\lambda > 0$, $\lambda = \alpha^2$, $\alpha > 0$

$r = \pm \alpha i$

$y = c_1 \cos \alpha x \quad + c_2 \sin \alpha x$

$y' = -c_1 \alpha \sin \alpha x \ + c_2 \alpha \cos \alpha x$

$y'(0) = c_2 \alpha = 0 \quad \rightarrow c_2 = 0$

$y'(1) = -c_1 \alpha \sin \alpha = 0 \qquad \alpha = n\pi$

$\lambda_n = (n\pi)^2 \quad y_n = c_n \cos(n\pi x) \quad n = 0, 1, 2, \ldots$

$\frac{g'}{3g} = -\lambda$

$\frac{g'}{g} = -3\lambda$

$\lambda = 0 \quad g' = 0 \quad g = c$

$\frac{g'}{g} = -3(n\pi)^2$

$\int \frac{g'}{g} = \int -3(n\pi)^2 \, dt$

$\ln|g| = -3(n\pi)^2 t + C$

$g = k e^{-3(n\pi)^2 t}$ $\quad n = 0$ covers $\lambda = 0$ case

$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x) e^{-3(n\pi)^2 t}$

$u(x,0) = \underline{\frac{a_0}{2} + \sum_{n=1}^{\infty}} a_n \cos n\pi x$

$u(x,t) = 6 + 5 \cos 2\pi x \, e^{-3(2\pi)^2 t} - 7 \cos 6\pi x \, e^{-3(6\pi)^2 t}$

2. (38 points)

a) $b_n = \frac{2}{3}\int_0^3 f(x)\sin\frac{n\pi x}{3}\,dx \quad n=1,2,\dots$

$$= \frac{2}{3}\int_0^2 (2-x)\sin\frac{n\pi x}{3}\,dx$$

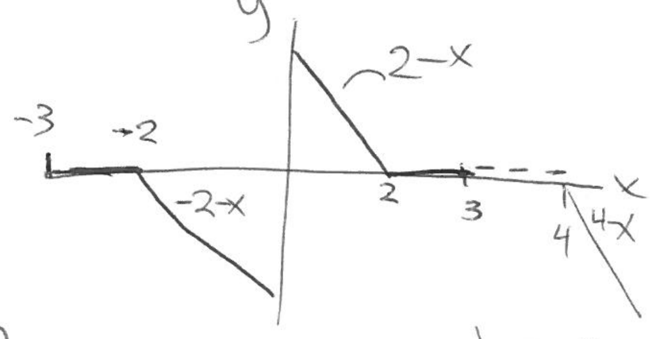
sign	u & its derivatives	dV & its antiderivatives
+	$2-x$	$\sin\frac{n\pi x}{3}$
−	-1	$-\frac{3}{n\pi}\cos\frac{n\pi x}{3}$
+	0	$\frac{-9}{n^2\pi^2}\sin\frac{n\pi x}{3}$

$$= \frac{2}{3}\left((2-x)\left(-\frac{3}{n\pi}\cos\frac{n\pi x}{3}\right) - \frac{9}{n^2\pi^2}\sin\frac{n\pi x}{3}\right)\Big|_0^2$$

$$= \frac{2}{3}\left(0 - \frac{9}{n^2\pi^2}\sin\frac{2n\pi}{3} - 2\left(\frac{-3}{n\pi}\right)\right) = \frac{-6}{n^2\pi^2}\sin\frac{2n\pi}{3} + \frac{4}{n\pi}$$

Fourier series $\rightarrow \sum_{n=1}^{\infty}\left(\frac{4}{n\pi} - \frac{6}{n^2\pi^2}\sin\frac{n\pi x}{3}\right)\sin\frac{n\pi x}{3}$

b) $f_o = \begin{cases} 2-x & 0<x<2 \\ 0 & 2<x<3 \\ -(2-(-x)) & -2<x<0 \\ 0 & -3<x<-2 \end{cases}$



for part c)

c) $-2+1.5 = \boxed{-.5}$, $\boxed{0}$, $\boxed{-1}$

3. (15 points)

$$r^2 + (\lambda - 2) = 0$$

$$r^2 = -(\lambda-2) = 2-\lambda$$

$$r = \pm\sqrt{2-\lambda}$$

$$2-\lambda < 0 \rightarrow 2-\lambda = -\alpha^2 \rightarrow r = \pm \alpha i$$

$$y = c_1 \cos \alpha x + c_2 \sin \alpha x$$

$$y(\pi) = c_1 \cos \alpha\pi + c_2 \sin \alpha \pi = 0$$

$$y' = -c_1 \alpha \sin \alpha x + c_2 \alpha \cos \alpha x$$

$$y'(0) = c_2 \alpha = 0$$

$$y(\pi) = c_1 \cos \alpha\pi = 0$$

$$\alpha\pi = \frac{(2n+1)\pi}{2}$$

$$2-\lambda = -\alpha^2 \qquad \lambda = 2 + \alpha^2$$

$$\boxed{\begin{array}{l} \lambda_n = 2 + \left(\frac{2n+1}{2}\right)^2 \quad n = 0, 1, 2, \ldots \\ y_n = c_n \cos\left(\frac{2n+1}{2} x\right) \end{array}}$$

$\leftarrow$ eigenvalue

$\leftarrow$ eigenfunctions