Sources & Nonhomogeneous Boundary Conditions

Goal: We want to be able to solve PDEs that are nonhomogeneous and/or have nonhomogeneous boundary conditions (BC)

Ex 1 Find a formal solution to

$u_t = u_{xx} \quad 0 < x < \pi, \; t > 0$

$u(0,t) = 0, \; u(\pi, t) = 3\pi, \; t > 0$

$u(x,0) = 0, \; 0 < x < \pi$

☆ Our d.e. is homogeneous (Recall: We can write a homogeneous 2nd order PDE in the form

$$A(x,t)u_{xx} + B(x,t)u_{xt} + C(x,t)u_{tt} + D(x,t)u_x + E(x,t)u_t + F(x,t)u = 0)$$

☆ 1 of our BC is nonhomogeneous $u(\pi, t) = 3\pi$ instead of zero

☆ Our BC are time independent so we assume that the temperature of the rod from $0 < x < \pi$ that this models will eventually be time independent as well. That is, $u(x,t) = \underbrace{v(x)}_{\text{steady-state solution}} + \underbrace{w(x,t)}_{\text{transient solution}}$

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If $u(x,t) = v(x) + w(x,t)$ then

$u_t = 0 + w_t$

$u_x = v'(x) + w_x$

$u_{xx} = v''(x) + w_{xx}$

☆ Plugging into our d.e.

$$\begin{cases} u_t = u_{xx} \rightarrow w_t = v''(x) + w_{xx} \\ u(0,t) = 0 \rightarrow v(0) + w(0,t) = 0 \\ u(\pi,t) = 3\pi \rightarrow v(\pi) + w(\pi,t) = 3\pi \end{cases}$$

☆ As $t \to \infty$ $w, w_t, w_{xx} \to 0$ so

$$w_t = v''(x) + w_{xx} \xrightarrow{\text{as } t\to\infty} \boxed{0 = v''(x)}$$

$$v(0) + w(0,t) = 0 \longrightarrow \boxed{v(0) = 0}$$

$$v(\pi) + w(\pi,t) = 3\pi \longrightarrow \boxed{v(\pi) = 3\pi}$$

☆ Solve the ODE we found to get the steady state solution

$v''(x) = 0 \rightarrow v'(x) = C \rightarrow v(x) = Cx + D$

$v(0) = 0 = C \cdot 0 + D$

$\rightarrow D = 0$

$v(x) = Cx$

$v(\pi) = 3\pi = C \cdot \pi$

$\rightarrow C = 3$

Steady-state solution : $v(x) = 3x$

☆ Plug $\boxed{\begin{matrix} v''(x) = 0 \\ v(0) = 0 \\ v(\pi) = 3\pi \end{matrix}}$ into $w_t = v''(x) + w_{xx}$

$v(0) + w(0,t) = 0$

$v(\pi) + w(\pi,t) = 3\pi$

to get $$W_t = 0 + W_{xx}$$
$$0 + w(0,t) = 0$$
$$\cancel{3\pi} + w(\pi,t) = \cancel{3\pi}$$

or $$W_t = W_{xx}$$
$$w(0,t) = 0$$
$$w(\pi,t) = 0$$

We've solved this PDE before! If we hadn't we would need to do separation of variables on this sub problem.

☆ In class we found the solution to

$$u_t = k u_{xx} \quad 0<x<L,\ t>0$$
$$u(x,0) = f(x) \quad 0<x<1$$
$$u(0,t) = 0 \quad t>0$$
$$u(L,t) = 0 \quad t>0$$

to be $$u(x,t) = \sum_{n=1}^{\infty} b_n \left(\sin\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 kt}$$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx \quad n=1,2,\ldots$$

so $$w(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{\pi}\right) e^{-\left(\frac{n\pi}{\pi}\right)^2 \cdot 1 \cdot t}$$
$$w(x,t) = \sum_{n=1}^{\infty} b_n \sin nx\, e^{-n^2 t}$$

☆ They gave us $u(x,0)=0$ at the begining

so $u(x,t) = v(x) + w(x,t) \rightarrow u(x,0) = v(x) + w(x,0) = 0$

$$w(x,0) = 0 - v(x)$$
$$= -3x$$

41 For our $w(x,t)$ sub-problem

$$w(x,0) = \underbrace{-3x}_{f(x) \text{ in our formula}}$$

$$w(x,t) = \sum_{n=1}^{\infty} b_n \sin nx \, e^{-n^2 t}$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} (-3x)\sin nx \, dx \qquad n = 1, 2, \ldots$$

Integration by parts: (tabular also works) $\quad u = -3x \quad v = -\frac{1}{n}\cos nx$

$du = -3dx \quad dv = \sin nx \, dx$

$$= \frac{2}{\pi}\left[uv - \int v\,du\right]$$

$$= \frac{2}{\pi}\left[(-3x)\left(-\frac{1}{n}\cos nx\right)\Big|_0^{\pi} - \int_0^{\pi} -\frac{1}{n}\cos nx(-3)\,dx\right]$$

$$= \frac{2}{\pi}\left[\left[\frac{3x}{n}\cos nx\right]_0^{\pi} - \frac{3}{n^2}\sin nx\Big|_0^{\pi}\right]$$

$$= \frac{2}{\pi}\left[\frac{3\pi}{n}\cos n\pi\right]$$

$\sin n\pi = 0$
$\sin 0 = 0$
$\frac{3 \cdot 0}{n}\cos 0 = 0$

$$= \frac{6}{n}\cos n\pi$$

$$= \frac{6}{n}(-1)^n$$

$$\underbrace{w(x,t) = \sum_{n=1}^{\infty} \frac{6}{n}(-1)^n \sin nx \, e^{-n^2 t}}$$

transient solution, note how we can see this going to 0 as $t \to \infty$

Ans: $\boxed{u(x,t) = 3x + \sum_{n=1}^{\infty} \frac{6}{n}(-1)^n \sin nx \, e^{-n^2 t}}$

Ex 2 Find a formal solution to the initial boundary value problem

$$u_t = 3u_{xx} + x \quad , \quad 0 < x < \pi \quad , \quad t > 0$$

$$u(0,t) = u(\pi,t) = 0 \quad , \quad t > 0$$

$$u(x,0) = \sin x \quad , \quad 0 < x < \pi$$

☆ Nonhomogeneous PDE! Our source term is x since this source term isn't a function of t we can solve this as before

$$u(x,t) = v(x) + w(x,t)$$

☆ Differentiate & plug in:

$$u_t = 3u_{xx} + x$$

$$0 + w_t = 3(v''(x) + w_{xx}) + x$$

$$\left.\begin{aligned} w_t &= 3v''(x) + 3w_{xx} + x \\ u(0,t) &= v(0) + w(0,t) = 0 \\ u(\pi,t) &= v(\pi) + w(\pi,t) = 0 \end{aligned}\right\} \quad \text{As } t \to \infty \quad w, w_t, w_{xx} \to 0$$

so $\quad 0 = 3v''(x) + x$

$$v(0) + 0 = 0$$

$$v(\pi) + 0 = 0$$

$$3v'' = -x$$

$$\boxed{\begin{aligned} v'' &= -\frac{x}{3} \\ v(0) &= 0 \\ v(\pi) &= 0 \end{aligned}}$$

☆ Solve this ODE

$$V'' = -\frac{x}{3} \rightarrow V' = -\frac{x^2}{6} + C \rightarrow V = \frac{-x^3}{18} + Cx + D$$

$$V(0) = \frac{-0^3}{18} + C\cdot 0 + D = 0 \rightarrow D = 0$$

$$V(\pi) = \underbrace{\frac{-\pi^3}{18} + C\pi}_{:(} = 0 \longrightarrow C = \frac{\pi^2}{18}$$

$$V(x) = \frac{-x^3}{18} + \frac{\pi^2}{18}x$$ } steady-state solution

☆ Plugging what we found into

$$W_t = 3V'' + 3W_{xx} + x$$

$$V(0) + w(0,t) = 0$$

$$V(\pi) + w(\pi, t) = 0$$

gives us

$$W_t = 3W_{xx}$$

$$w(0,t) = 0$$

$$w(\pi, t) = 0$$

☆ Plug into our in class result (page 3 of this worksheet) with $k=3$, $L=\pi$ to get

$$w(x,t) = \sum_{n=1}^{\infty} b_n \sin nx \, e^{-3n^2 t}$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx \qquad n = 1, 2, \ldots$$

↑ $w(x,0)$

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$$u(x,0) = v(x) + w(x,0) = \underbrace{\sin x}_{\text{given}}$$

$$w(x,0) = \sin x - v(x)$$

$$b_n = \frac{2}{\pi}\int_0^\pi \left(\sin x - \left(-\frac{x^3}{18} + \frac{\pi^2}{18}x\right)\right)\sin nx\,dx$$

$$= \underbrace{\frac{2}{\pi}\int_0^\pi \sin x \sin nx\,dx}_{} + \frac{2}{\pi}\int_0^\pi \underbrace{\left(\frac{x^3}{18} - \frac{\pi^2}{18}x\right)}_{}\sin nx$$

In class we saw that $\int_0^\pi \sin nx \sin mx\,dx$ is 0 if $m \neq n$ & $\pi/2$ if $m=n$ so if $n=1$ this is $\pi/2$ if $n \neq 1$ this is 0

I distributed the negative

$$\int_0^\pi \left(\frac{x^3}{18} - \frac{\pi^2}{18}x\right)\sin nx\,dx$$

Tabular integration by parts!

does this exclamation make it seem like we are excited to do this?

Sign	u & its derivatives	dV & its antiderivatives
+	$x^3/18 - \pi^2/18\, x$	$\sin nx$
−	$\frac{3x^2}{18} - \frac{\pi^2}{18}$	$-\frac{1}{n}\cos nx$
+	$6x/18$	$-\frac{1}{n^2}\sin nx$
−	$6/18$	$\frac{1}{n^3}\cos nx$
+	0	$\frac{1}{n^4}\sin nx$

$$= \left(\frac{x^3}{18} - \frac{\pi^2}{18}x\right)\left(-\frac{1}{n}\cos nx\right) - \left(\frac{3x^2}{18} - \frac{\pi^2}{18}\right)\left(-\frac{1}{n^2}\sin nx\right) + \left(\frac{6x}{18}\right)\left(\frac{1}{n^3}\cos nx\right)$$

$$-\frac{6}{18}\left(\frac{1}{n^4}\sin nx\right)\Bigg|_0^\pi$$

$$= \cancel{0\left(-\frac{1}{n}\cos n\pi\right)} - \cancel{\left(\frac{3\pi^2}{18} - \frac{\pi^2}{18}\right)\left(-\frac{1}{n^2}\sin n\pi\right)} + \frac{6\pi}{18}\left(\frac{1}{n^3}\cos n\pi\right) - \frac{6}{18}\cancel{\left(\frac{1}{n^4}\sin n\pi\right)}^{0}$$

$$-[0] = \frac{\pi}{3n^3}\cos n\pi = \frac{\pi}{3n^3}(-1)^n$$

(I actually liked this part)

$$b_n = \frac{2}{\pi}\begin{pmatrix} 0 \text{ if } n \neq 1 \\ \pi/2 \text{ if } n = 1 \end{pmatrix} + \frac{2}{\pi}\left(\frac{\pi}{3n^3}(-1)^n\right)$$

$$b_1 = \frac{2}{\pi}\cdot\frac{\pi}{2} + \frac{2}{3\cdot 1^3}(-1)^1$$

$$b_1 = 1 - \frac{2}{3} = \frac{1}{3}$$

If $n \neq 1$, $b_n = \frac{2}{3n^3}(-1)^n$

$$u(x,t) = v(x) + w(x,t)$$

$$\boxed{u(x,t) = \underbrace{\frac{-x^3}{18} + \frac{\pi^2}{18}x}_{\text{steady-state solution}} \quad \underbrace{+\frac{1}{3}\sin(1\cdot x)e^{-3t} + \sum_{n=2}^{\infty}\frac{2}{3n^3}(-1)^n \sin nx\, e^{-3n^2 t}}_{\text{transient solution}}}$$

Ex 3 Consider the 1D heat equation in a rod of length L with diffusion constant k. Suppose the left endpoint is fixed at $100°$, while the right endpoint is insulated, & the initial temperature in the rod is

$f(x) = 1000x^3(1-x) + 100$

a) Set up the initial boundary problem modeling this scenario

$$\begin{aligned} \star\quad & u_t = k u_{xx}, && 0<x<L,\ t>0 \\ & u(0,t) = 100, && t>0 \\ & u_x(L,t) = 0, && t>0 \\ & u(x,0) = 1000x^3(1-x) + 100, && 0<x<L \end{aligned}$$

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b) Set up & solve the steady-state problem for part a)

☆ $u(x,t) = v(x) + w(x,t)$

$$\left.\begin{aligned} w_t &= k(v''(x) + w_{xx}) \\ u(0,t) &= 100 = v(0) + w(0,t) \\ u_x(L,t) &= 0 = v'(L) + w_x(L,t) \end{aligned}\right\} (☆☆)$$

As $t \to \infty$ (☆☆) goes to $0 = kv''(x)$

$v(0) = 100$

$v'(L) = 0$

$v'' = 0 \rightarrow v' = C \rightarrow v = Cx + D$

$v(0) = 100 = C \cdot 0 + D \quad D = 100$

$v(x) = Cx + 100$

$v'(x) = C$

$v'(L) = C = 0$

$$\boxed{v(x) = 100}$$

c) set up & solve the transient problem for a)

☆ (☆☆) with $v(x)=100$, $v(0)=100$, $v'(L)=0$ plugged in

$w_t = k w_{xx}$

$w(0,t) = 0$

$w_x(L,t) = 0$

10 ☆ I don't think we've solved this one unfortunately.

$w_t = k w_{xx}$

$w(0,t) = w_x(L,t) = 0$

☆ Separation of variables

$w(x,t) = X(x)T(t)$ $\quad$ $w(0,t) = X(0)T(t) = 0$

$\rightarrow X(0) = 0$

$w_t = k w_{xx}$ $\quad$ $w_x(L,t) = X'(L)T(t) = 0$

$XT' = kX''T$ $\quad$ $\rightarrow X'(L) = 0$

$$\frac{X''}{X} = \frac{T'}{kT} = -\lambda$$

$$X'' = -\lambda X$$

$$X'' + \lambda X = 0$$

$$r^2 + \lambda = 0$$

$$r = \pm\sqrt{-\lambda}$$

Cases:

$\lambda < 0$, $\lambda = -\alpha^2$, $\alpha > 0$, $r = \pm\alpha$

$$X = c_1 \cosh \alpha x + c_2 \sinh \alpha x$$

$$X(0) = c_1 \cosh 0 + c_2 \sinh 0 = c_1 = 0$$

$$X' = c_2 \alpha \cosh \alpha x$$

$X'(L) = c_2 \alpha$ $\quad$ $h \alpha L = 0 \rightarrow c_2 = 0$ trivial solution!

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$\lambda = 0, \ r = 0$

$X = c_1 + c_2 x$

$X(0) = c_1 = 0$

$X' = c_2$

$X'(L) = c_2 = 0 \rightarrow$ trivial sol,

$\lambda > 0, \ \lambda = \alpha^2, \ \alpha > 0, \ r = \pm \alpha i$

$X = c_1 \cos \alpha x + c_2 \sin \alpha x$

$X(0) = c_1 = 0$

$X' = c_2 \alpha \cos \alpha x$

$X'(L) = c_2 \alpha \cos \alpha L = 0$

$c_2 \neq 0$, otherwise we'd have the trivial solution

so $\cos \alpha L = 0$

$\alpha L = \left(\frac{(2n+1)\pi}{2}\right) \quad n = 0, 1, 2, \ldots$

$\rightarrow \alpha = \frac{(2n+1)\pi}{2L}$

$X = c_2 \sin\left(\left(\frac{(2n+1)\pi}{2L}\right)x\right) n = 0, 1, 2, \ldots$

c_n is good here as well

$\rightarrow \lambda_n = \left[\frac{(2n+1)\pi}{2L}\right]^2$

$n = 0, 1, 2, \ldots$

$\frac{T'}{kT} = -\lambda \rightarrow T' + \lambda k T = 0$

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$$\lambda_n = \left[\frac{(2n+1)\pi}{2L}\right]^2$$

$$T' + \left[\frac{(2n+1)\pi}{2L}\right]^2 kT = 0 \qquad n = 0, 1, 2, \ldots$$

$n=0$ doesn't make this $T'=0$ so we don't need to do that case separately

$$\frac{dT}{dt} = -\left[\frac{(2n+1)\pi}{2L}\right]^2 kT$$

$$\int \frac{dT}{T} = \int -\left[\frac{(2n+1)\pi}{2L}\right]^2 k \, dt$$

$$\ln|T| = -\left[\frac{(2n+1)\pi}{2L}\right]^2 kt + C$$

$$T = a e^{-\left[\frac{(2n+1)\pi}{2L}\right]^2 kt}$$

$$w(x,t) = X(x)T(t)$$

$$w(x,t) = \sum_{n=0}^{\infty} b_n \sin\left(\frac{(2n+1)\pi x}{2L}\right) e^{-\left(\frac{(2n+1)\pi}{2L}\right)^2 kt}, \quad n = 0, 1, 2, \ldots$$

Note: $\frac{(2n+1)\pi}{2L} = \left\{\frac{\pi}{2L}, \frac{3\pi}{2L}, \frac{5\pi}{2L}, \ldots\right\}$ for $n = 0, 1, 2, \ldots$

is equivalent to $\frac{(2n-1)\pi}{2L} = \left\{\frac{\pi}{2L}, \frac{3\pi}{2L}, \frac{5\pi}{2L}\right\}$ for $n = 1, 2, \ldots$

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$$w(x,t)=\sum_{n=1}^{\infty} b_n \sin\left(\frac{(2n-1)\pi x}{2L}\right) e^{-\left(\frac{(2n-1)\pi}{2L}\right)^2 kt} \Big\}\text{ transient solution}$$

$$w(x,0)=\sum_{n=1}^{\infty} b_n \sin\left(\frac{(2n-1)\pi x}{2L}\right)$$

$$u(x,0)=1000x^3(1-x)+100$$

$$u(x,t)=v(x)+w(x,t)$$

$$u(x,0)=v(x)+w(x,0)=u(x,0)$$

$$w(x,0)=u(x,0)-v(x)$$

$$w(x,0)=1000x^3(1-x)+100-(100)$$

$$w(x,0)=1000x^3(1-x)$$

Recall:

$$\boxed{f(x)=\sum_{n=1}^{\infty} c_n \underbrace{f_n(x)}_{\text{orthogonal functions}} \qquad c_n=\frac{(f,f_n)}{\|f_n\|^2}}$$

so $$b_n=\frac{\left(1000x^3(1-x), \sin\left(\frac{(2n-1)\pi x}{2L}\right)\right)}{\left\|\sin\left(\frac{(2n-1)\pi x}{2L}\right)\right\|^2}$$

$$b_n=\frac{\int_0^L 1000x^3(1-x)\sin\left(\frac{(2n-1)\pi x}{2L}\right)dx}{\int_0^L \sin^2\left(\frac{(2n-1)\pi x}{2L}\right)dx}\Bigg\}\text{ other part of the transient solution}$$

$n=1,2,3,\ldots$

14 d) Solve the problem in part a)

$$u(x,t) = 100 + \sum_{n=1}^{\infty} b_n \sin\left(\frac{(2n-1)\pi x}{2L}\right) e^{-\left(\frac{(2n-1)\pi}{2L}\right)^2 kt}$$

where $b_n = \dfrac{\int_0^L 1000x^3(1-x)\sin\left(\frac{(2n-1)\pi x}{2L}\right)dx}{\int_0^L \sin^2\left(\frac{(2n-1)\pi x}{2L}\right)dx}$

☆ Aspects of this problem are ok but for these problems, it either needs to yield a result we've done in class like $\begin{cases} w_t = k w_{xx} \\ w(0,t) = w(L,t) = 0 \\ w(x,0) = f(x) \end{cases}$

or it needs to be given.

Ex 4 Final a formal solution to

$u_t = 3u_{xx} + 5, \quad 0 < x < \pi, \quad t > 0$

$u(0,t) = u(\pi,t) = 1, \quad t > 0$

$u(x,0) = 1, \quad 0 < x < \pi$

☆ $u(x,t) = v(x) + w(x,t)$

$$\left.\begin{aligned} w_t &= 3v'' + 3w_{xx} + 5 \\ u(0,t) &= v(0) + w(0,t) = 1 \\ u(\pi,t) &= v(\pi) + w(\pi,t) = 1 \end{aligned}\right\} \text{as } t \to \infty \quad \begin{aligned} 0 &= 3v'' + 5 \\ v(0) &= 1 \\ v(\pi) &= 1 \end{aligned}$$

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$$v'' = -\frac{5}{3} \longrightarrow v' = -\frac{5}{3}x + C \rightarrow v = -\frac{5}{6}x^2 + Cx + D$$

$$v(0) = D = 1$$

$$v = -\frac{5}{6}x^2 + Cx + 1$$

$$v(\pi) = -\frac{5}{6}\cdot\pi^2 + C\pi + 1 = 1$$

$$C = \frac{5}{6}\pi$$

$$v(x) = -\frac{5}{6}x^2 + \frac{5}{6}\pi x + 1 \quad \Big\} \text{ steady-state solution}$$

☆ Plug into

$$\left.\begin{aligned} w_t &= 3v'' + 3w_{xx} + 5 \\ v(0) + w(0,t) &= 1 \\ v(\pi) + w(\pi,t) &= 1 \end{aligned}\right\} \quad \begin{aligned} w_t &= 3w_{xx} \\ w(0,t) &= 0 \\ w(\pi,t) &= 0 \end{aligned}$$

$$w(x,t) = \sum_{n=1}^{\infty} b_n \sin nx \, e^{-3n^2 t} \qquad n = 1, 2, \ldots$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} w(x,0)\sin nx \, dx$$

$$u(x,0) = 1 = v(x) + w(x,0)$$

$$w(x,0) = 1 - v(x) = \frac{5}{6}x^2 - \frac{5}{6}\pi x$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} \left(\frac{5}{6}x^2 - \frac{5}{6}\pi x\right)\sin nx \, dx$$

16b)

Tablar!

Sign	u & its der.	dV & its anti
+	$\frac{5}{6}x^2 - \frac{5}{6}\pi x$	$\sin nx$
−	$\frac{10}{6}x - \frac{5}{6}\pi$	$-\frac{1}{n}\cos nx$
+	$\frac{10}{6}$	$-\frac{1}{n^2}\sin nx$
−	0	$\frac{+}{n^3}\cos nx$

$$b_n = \frac{2}{\pi}\left[\left(\frac{5}{6}x^2 - \frac{5}{6}\pi x\right)\left(\frac{-1}{n}\cos nx\right) + \left(\frac{10}{6}x - \frac{10}{6}\pi\right)\left(\frac{1}{n^2}\sin nx\right) + \frac{10}{6}\cdot\frac{1}{n^3}\cos nx\right]_0^{\pi}$$

$$b_n = \frac{2}{\pi}\left[\frac{10}{6}\cdot\frac{1}{n^3}\cos \pi n - \frac{10}{6}\cdot\frac{1}{n^3}\cos 0\right]$$

$$= \frac{20}{6n^3\pi}\left[\cos n\pi - 1\right]$$

$$= \frac{10}{3n^3\pi}\left[(-1)^n - 1\right]$$

$$u(x,t) = V(x) + w(x,t)$$

$$\boxed{u(x,t) = -\frac{5}{6}x^2 + \frac{5\pi}{6}x + 1 + \underbrace{\sum_{n=1}^{\infty}\frac{10}{3n^3\pi}\left[(-1)^n - 1\right]\sin nx\, e^{-3n^2 t}}}$$

↑ good enough

If $n = 1, 3, 5, \ldots$ $(-1)^n - 1 = -2$
If $n = 2, 4, 6, \ldots$ $(-1)^n - 1 = 0$

$$u(x,t) = -\frac{5}{6}x^2 + \frac{5\pi}{6}x + 1 + \underbrace{\frac{10}{3}\sum_{k=0}^{\infty}\frac{(-2)}{\pi}\cdot\frac{1}{(2k+1)^3}\sin((2k+1)x)\, e^{-3(2k+1)^2 t}}_{\text{fancy}}$$

Ex 5 Find a formal solution to the vibrating spring problem where

$$u_{tt} = 4u_{xx}, \quad 0 < x < 1, \quad t > 0$$

$u(0,t) = 0 \qquad t > 0$

$u(1,t) = 12 \qquad t > 0$

$u(x,0) = 6 + 12x$

$u_t(x,0) = 0$

☆ We are going to approach this as before in the heat equation but now $v(x)$ is the time-independent equilibrium state (technically the undamped wave equation doesn't have a steady-state)

$u(x,t) = v(x) + w(x,t)$

$$\left.\begin{aligned} w_{tt} &= 4(v'' + w_{xx}) \\ u(0,t) &= v(0) + w(0,t) = 0 \\ u(1,t) &= v(1) + w(1,t) = 12 \end{aligned}\right\} \begin{aligned} &\text{as } t \to \infty \\ &0 = 4v'' \\ &v(0) = 0 \\ &v(1) = 12 \end{aligned}$$

$v'' = 0 \to v' = C \to v = Cx + D$

$v(0) = D = 0 \quad v(1) = 12 = C \cdot 1$

$v(x) = 12x$

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Plugging back in:

$$w_{tt} = 4w_{xx}$$
$$w(0,t) = 0$$
$$w(1,t) = 0$$

★ Use the result from in class with $c=2$, $L=1$

$$w(x,t) = \sum_{n=1}^{\infty} \left(a_n \cos(2n\pi t) + b_n \sin(2n\pi t)\right) \sin n\pi x$$

$$w(x,0) = \sum_{n=1}^{\infty} a_n \cdot 1 \sin n\pi x$$

$$w_t(x,t) = \sum_{n=1}^{\infty} 2n\pi \left[-a_n \sin(2n\pi t) + b_n \cos(2n\pi t)\right] \sin n\pi x$$

$$w_t(x,0) = \sum_{n=1}^{\infty} 2n\pi b_n \cdot 1 \sin n\pi x$$

$$u(x,t) = v(x) + w(x,t)$$
$$u(x,t) = 12x + w(x,t)$$
$$u(x,0) = 12x + w(x,0)$$
$$u_t(x,t) = 0 + w_t(x,t)$$
$$u_t(x,0) = w_t(x,0) = 0 = \sum_{n=1}^{\infty} 2n\pi b_n \sin n\pi x \rightarrow$$

given in this problem

$b_n = 0$

$$w(x,0) = u(x,0) - 12x$$
$$= 6 + 12x - 12x$$
$$= 6$$

$$6 = \sum_{n=1}^{\infty} a_n \sin n\pi x$$

$$f(x) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{L}$$

$$\rightarrow a_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}$$

So $$a_n = \frac{2}{1} \int_0^1 6 \sin n\pi x \, dx$$

$$= 12 \left(\frac{-1}{n\pi} \cos n\pi x \right)\Big|_0^1$$

$$= \frac{-12}{n\pi} \left[\cos \pi n - 1 \right]$$

$$= \frac{-12}{n\pi} \left[(-1)^n - 1 \right]$$

$$= \frac{12}{n\pi} \left[(-1)^{n+1} + 1 \right]$$

$$\boxed{u(x,t) = 12x + \sum_{n=1}^{\infty} \frac{12}{n\pi} \underbrace{\left[(-1)^{n+1} + 1 \right]} \cos 2n\pi t \sin n\pi x}$$

could mess w/ this or leave it. I opt for leaving it.

Ex 1, Ex 2, Ex 4 & sort of Ex 5 from Fundamentals of Differential Equations by Nagle, Saff, & Snider

Ex 3 from Intro to Applied Partial Differential Equations by Davis