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Convolution

Def: Let $f(t)$ and $g(t)$ be piecewise continuous on $[0,\infty)$ The <u>convolution</u> of $f(t)$ & $g(t)$, denoted $f*g$, is

$$f(t)*g(t) = \int_0^t f(t-v)\,g(v)\,dv$$

<u>Convolution</u> <u>Theorem</u>: Let $f(t)$ & $g(t)$ be piecewise continuous on $[0,\infty)$ & of exponential order α $\Big\}$ conditions so that $\mathcal{L}\{f\}$ & $\mathcal{L}\{g\}$ exist

Let $F(s) = \mathcal{L}\{f\}$ & $G(s) = \mathcal{L}\{g\}$, then

$$\mathcal{L}\{f*g\} = F(s)\,G(s)$$

or

$$\mathcal{L}^{-1}\{F(s)\,G(s)\} = f*g$$

Pf: Recall: $\mathcal{L}\{f\} = \int_0^\infty e^{-st} f(t)\,dt$ so

$$\mathcal{L}\{f*g\} = \int_0^\infty e^{-st}(f*g)\,dt \overset{\text{def of convolution}}{=}$$

$$= \int_0^\infty e^{-st}\left(\int_0^t f(t-v)\,g(v)\,dv\right)dt$$

2) $\mathcal{L}\{f*g\} = \int_0^\infty e^{-st}\left(\int_0^t f(t-v)\,g(v)\,dv\right)dt$

Recall: Unit step functions:

$$u(t-a) = \begin{cases} 0 & t<a \\ 1 & t\geq a \end{cases}$$

so $$u(t-v) = \begin{cases} 0 & t<v \\ 1 & t\geq v \end{cases}$$

We have $\underbrace{\int_0^t f(t-v)\,g(v)\,dv}_{0\leq v\leq t}$

so $\int_0^t f(t-v)g(v)\,dv = \int_0^t f(t-v)g(v)u(t-v)\,dv$

$+ \int_t^\infty f(t-v)g(v)u(t-v)\,dv$

$= \int_0^\infty f(t-v)\,g(v)\,u(t-v)\,dv$

$t\leq v<\infty \rightarrow \int_t^\infty f(t-v)g(v)u(t-v)\,dv = 0$

Plugging in to the top equation:

$\mathcal{L}\{f*g\} = \int_0^\infty e^{-st}\int_0^\infty f(t-v)g(v)u(t-v)\,dv\,dt$

We can switch the order of integration by Fubini's theorem

3| $\mathcal{L}\{f*g\} = \int_0^\infty \int_0^\infty e^{-st} f(t-v)g(v)u(t-v)\,dt\,dv$

$$= \int_0^\infty g(v) \underbrace{\int_0^\infty e^{-st}\big(f(t-v)u(t-v)\big)dt}_{\text{def of Laplace transform}}\, dv$$

$$= \int_0^\infty g(v)\,\mathcal{L}\{f(t-v)u(t-v)\}\,dv$$

In the Unit Step Function section we showed $\mathcal{L}\{f(t-v)u(t-v)\} = e^{-vs}\mathcal{L}\{f\}$

so

$$\mathcal{L}\{f*g\} = \int_0^\infty g(v)e^{-vs}\underbrace{\mathcal{L}\{f\}}_{\substack{\text{function of } s\text{, constant} \\ \text{w.r.t } v}}\,dv$$

$$= \underbrace{\int_0^\infty g(v)e^{-vs}\,dv}_{\substack{\text{def of Laplace} \\ v \text{ is just a} \\ \text{dummy variable}}}\,\mathcal{L}\{f\}$$

$$= \mathcal{L}\{g\}\,\mathcal{L}\{f\}\ \checkmark$$

Examples from <u>Fundamentals of Differential Equations</u> by Nagle, Saff & Snider

$\boxed{\text{Ex 1}}$ Use the convolution theorem to get a formula for the solution assuming $g(t)$ is piecewise continuous & of exponential order on $[0,\infty)$

$$y''-2y'+y=g(t) \quad y(0)=-1,\ y'(0)=1$$

Like usual, I'll use $L=\mathcal{L}\{y\}$.
Using our tables, the left becomes

$$s^2L-sy(0)-y'(0)-2[sL-y(0)]+L=\mathcal{L}\{g\}$$

$$(s^2-2s+1)L+s-1-2=\mathcal{L}\{g\}$$

$$L=\frac{\mathcal{L}\{g\}+3-s}{s^2-2s+1}$$

$$=\mathcal{L}\{g\}\frac{1}{s^2-2s+1}+\frac{3-s}{s^2-2s+1}$$

$$=\mathcal{L}\{g\}\frac{1}{(s-1)^2}+\frac{3-s}{(s-1)^2}$$

$$\frac{3-s}{(s-1)^2}=\frac{A}{s-1}+\frac{B}{(s-1)^2} \qquad A(s-1)+B=3-s$$

$$As-A+B \qquad A=-1,\ B=2$$

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$$L = \mathcal{L}\{g\}\frac{1}{(s-1)^2} - \frac{1}{s-1} + \frac{2}{(s-1)^2}$$

$$y = \mathcal{L}^{-1}\left\{\mathcal{L}\{g\}\frac{1}{(s-1)^2}\right\} - e^t + 2te^t$$

$$= \mathcal{L}^{-1}\left\{G(s)\frac{1}{(s-1)^2}\right\} - e^t + 2te^t$$

$$= g(t) * te^t - e^t + 2te^t$$

$$y = \int_0^t g(t-v)\underbrace{ve^v}\,dv - e^t + 2te^t$$

switch ts to
Vs

$f * g = g * f$ so we could have written this differently as

$$y = \int_0^t g(v)(t-v)e^{t-v}\,dv - e^t + 2te^t$$

$\boxed{\text{Ex 2}}$ Find the inverse Laplace of the following using the convolution theorem

$$\mathcal{L}^{-1}\left\{\frac{14}{(s+2)(s-5)}\right\} = \mathcal{L}^{-1}\left\{\frac{14}{s+2}\right\} * \mathcal{L}^{-1}\left\{\frac{1}{s-5}\right\}$$

$$= 14e^{-2t} * e^{5t} =$$

$$\int_0^t 14e^{-2(t-v)}e^{5v}\,dv = \int_0^t 14e^{-2t}e^{2v}e^{5v}\,dv$$

$$= \int_0^t 14e^{-2t}e^{7v}\,dv = 14e^{-2t}\frac{1}{7}e^{7v}\Big|_0^t$$

$$= 2e^{-2t}e^{7t} - 14e^{-2t}\frac{1}{7}e^0$$

$$= 2e^{5t} - 2e^{-2t}$$

Ex 3 Find the Laplace transform of $f(t) = \int_0^t e^v \sin(t-v)\,dv$

Recall: $f * g = \int_0^t f(t-v)\,g(v)\,dv$

Not the $f(t)$ mentioned directly above. This is just the definition of convolution

so $f(t) = \int_0^t e^v \sin(t-v)\,dv =$

$$\int_0^t \sin(t-v)\,e^v\,dv$$

$$= \sin t * e^t$$

$$\mathcal{L}\{f\} = \mathcal{L}\{\sin t * e^t\}$$

convolution theorem tells us this is

$$= \mathcal{L}\{\sin t\}\,\mathcal{L}\{e^t\}$$

$$= \frac{1}{s^2+1}\,\frac{1}{s-1}$$

$$= \boxed{\frac{1}{(s^2+1)(s-1)}}$$

$\boxed{\text{Ex } 4}$ Find $y(t)$

$$y(t) + 3\int_0^t y(v)\sin(t-v)\,dv = t$$

$$y + 3\int_0^t y(v)\sin(t-v)\,dv = t$$

$$y + 3\,y(t) * \sin t = t$$

$$\mathcal{L}\{y\} + 3\mathcal{L}\{y(t) * \sin t\} = \mathcal{L}\{t\}$$

$$L + 3L\frac{1}{s^2+1} = \frac{1}{s^2}$$

$$L\left(1 + \frac{3}{s^2+1}\right) = \frac{1}{s^2}$$

$$L\left(\frac{s^2+1+3}{s^2+1}\right) = \frac{1}{s^2}$$

$$L = \frac{1}{s^2}\,\frac{(s^2+1)}{s^2+4}$$

$$= \frac{(s^2+1)}{s^2(s^2+4)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+2D}{s^2+4}$$

$$As(s^2+4) + B(s^2+4) + (Cs+2D)s^2 = s^2+1$$

$$\underbrace{As^3}_{} + \underline{4As} + \underline{Bs^2} + 4B + \underbrace{Cs^3}_{} + \underline{2Ds^2} = \underline{s^2} + 1$$

$A + C = 0$, $\cancel{}$, $B + 2D = 1$, $A = 0$ $\quad 4B = 1$

$C = 0$ $\quad B = 1/4$

$D = 3/8$

$$L = \frac{1/4}{s^2} + 3/8\,\frac{2}{s^2+4}$$

$$y = \frac{1}{4}t + 3/8\,\sin 2t$$