Matrix Exponential Function

Recall: the Maclaurin series for $f(t) = e^{at} = \sum_{n=0}^{\infty} \frac{(at)^n}{n!} = 1 + at + \frac{a^2t^2}{2!} + \dots$

Let $A_{n \times n}$ be our coefficient matrix.

e^{At} is defined to be

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \dots$$

matrix version of 1

Note: $\frac{d}{dt}\left(e^{At}\right) = \frac{d}{dt}\left(I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \dots\right)$

$$= A + \frac{2A^2t}{2!} + \frac{3A^3t^2}{3!} + \dots$$

$\frac{3}{3!} = \frac{3}{3 \cdot 2 \cdot 1} = \frac{1}{2!}$

$$= A\left(I + At + \frac{A^2t^2}{2!} + \dots\right)$$

so $\frac{d}{dt}\left(e^{At}\right) = A e^{At}$

Note: $\vec{x} = e^{At}$ solves $\vec{x}\,' = A\vec{x}$

$\vec{x}\,' = Ae^{At}$ plugging in

$$\underbrace{Ae^{At}}_{\vec{x}\,'} = A\underbrace{e^{At}}_{\vec{x}} \quad \checkmark$$

The general solution to $\vec{x}\,' = A\vec{x}$ is $\vec{x} = e^{At}\vec{c}$ ← constant vector

To find our general solution to $\vec{x}\,' = A\vec{x}$ we can do the following:

a) Use the characteristic equation $|A - rI| = 0$ to find all eigenvalues & their multiplicities

b) For each eigenvalue r_i w/ multiplicity m_i use $(A - r_i I)^{m_i} \vec{u} = \vec{0}$ to find generalized eigenvectors $\vec{u}$

c) For each of these eigenvectors form $\vec{x}(t) = e^{rt}\left[\vec{u} + t(A-rI)\vec{u} + \frac{t^2(A-rI)^2}{2!} + \frac{t^3(A-rI)^3}{3!} + \cdots\right]$

d) The general solution is then $\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2 + \cdots + c_n\vec{x}_n$

You can stop when you get to $(A-rI)^{m_i}$

Examples from Fundamentals of Differential Equations by Nagle, Saff, & Snider

Ex 1: Find the general solution to $\vec{x}' = \begin{bmatrix} 3 & -2 \\ 0 & 3 \end{bmatrix} \vec{x}$

a) $|A - rI| = \begin{vmatrix} 3-r & -2 \\ 0 & 3-r \end{vmatrix} = (3-r)^2 = 0$

$r_1 = r_2 = 3$, multiplicity $= 2$

Note: $(A-3I)\vec{u} = \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix} \vec{u} = \vec{0}$

won't give us 2 linearly independent eigenvectors. This technique is good in these scenarios

b) $(A - r_i I)^{m_i} \vec{u} = \vec{0}$

$$(A - 3I)^2 \vec{u} = \vec{0}$$

$$\begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix}^2 \vec{u} = \vec{0}$$

$$\begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix} \vec{u} = \vec{0}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \vec{u} = \vec{0}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} u_a \\ u_b \end{bmatrix} = \vec{0}$$

$$\vec{u_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \vec{u_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Note: After we find $(A - r_i I)^{m_i}\vec{u} = \vec{0}$ we find our eigenvectors as before

c) $$\vec{x_1}(t) = e^{3t}\left[\begin{bmatrix} 1 \\ 0 \end{bmatrix} + t(A-3I)\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \underbrace{\frac{t^2(A-3I)^2}{2!}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \vec{0}}_{\text{all later entries will be } \vec{0}}\right]$$

$$\vec{x_1}(t) = e^{3t}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \cancel{\frac{t^2\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}}{2!}}\right)$$

$$= e^{3t}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right)$$

$$= e^{3t}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\vec{x}_2(t) = e^{3t}\left(\begin{bmatrix}0\\1\end{bmatrix} + t\begin{bmatrix}0 & -2\\0 & 0\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} + \cancel{t^2\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}}\right)$$

$$= e^{3t}\left(\begin{bmatrix}0\\1\end{bmatrix} + t\begin{bmatrix}-2\\0\end{bmatrix}\right)$$

$$= e^{3t}\begin{bmatrix}-2t\\1\end{bmatrix}$$

Our solution is $\vec{x} = c_1\vec{x}_1 + c_2 x_2$

$$\boxed{\vec{x} = c_1 e^{3t}\begin{bmatrix}1\\0\end{bmatrix} + c_2 e^{3t}\begin{bmatrix}-2t\\1\end{bmatrix}}$$

Ex 2: Find the general solution to $\vec{x}\,' = \begin{bmatrix}1 & -1\\1 & 3\end{bmatrix}\vec{x}$

a) $|A - rI| = \begin{vmatrix}1-r & -1\\1 & 3-r\end{vmatrix} = (1-r)(3-r)+1 = 0$

$r^2 - 4r + 3 + 1 = 0$

$r^2 - 4r + 4 = 0 \rightarrow (r-2)^2 = 0 \quad r_1 = r_2 = 2$

mult $= 2$

As with the previous problem this repeated eigenvalue won't work with our 9.5 techniques

b) $(A-2I)^2 \vec{u} = \vec{0}$

$$A-2I = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$(A-2I)^2 = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} (1-1) & (1-1) \\ (-1+1) & (-1+1) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \vec{u} = \vec{0}$$

$$\vec{u_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \vec{u_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

c) $\vec{x_1}(t) = e^{2t}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + t(A-2I)\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \vec{0} \right)$

$$= e^{2t}\left(\begin{bmatrix} \text{[redacted]} \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$$

$$= e^{2t}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right)$$

$$\vec{x_2}(t) = e^{2t}\left(\begin{bmatrix}0\\1\end{bmatrix} + t\begin{bmatrix}-1 & -1\\1 & 1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} + \vec{0}\right)$$

$$= e^{2t}\left(\begin{bmatrix}0\\1\end{bmatrix} + t\begin{bmatrix}-1\\1\end{bmatrix}\right)$$

$$\boxed{\vec{x} = c_1 e^{2t}\begin{bmatrix}1-t\\t\end{bmatrix} + c_2 e^{2t}\begin{bmatrix}-t\\1+t\end{bmatrix}}$$

Ex 3: $\vec{x}' = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 2 \end{bmatrix} \vec{x}$

a) $|A - rI| = \begin{vmatrix} 2-r & 1 & 3 \\ 0 & 2-r & -1 \\ 0 & 0 & 2-r \end{vmatrix} = (2-r)^3 = 0$

$r_1 = r_2 = r_3 = 2$, multiplicity $= 3$

b) $(A - 2I)^3 \vec{u} = \vec{0}$

$$\begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}^3 \vec{u} = \vec{0}$$

$$\underbrace{\begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}}_{(A-2I)^2} \begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \vec{u} = \vec{0}$$

$$\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \vec{u} = \vec{0}$$

$\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\vec{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $\vec{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_a \\ u_b \\ u_c \end{bmatrix} = \vec{0}$$

c) $$\vec{x}_1(t) = e^{2t}\left[\begin{bmatrix}1\\0\\0\end{bmatrix} + t(A-2I)\begin{bmatrix}1\\0\\0\end{bmatrix} + \frac{t^2(A-2I)^2}{2!} + \vec{0}\right]$$

$$= e^{2t}\left[\begin{bmatrix}1\\0\\0\end{bmatrix} + t\begin{bmatrix}0&1&3\\0&0&-1\\0&0&0\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix} + \frac{t^2}{2!}\begin{bmatrix}0&0&-1\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix}\right]$$

$$\vec{x}_1 = e^{2t}\left(\begin{bmatrix}1\\0\\0\end{bmatrix} + t\begin{bmatrix}0\\0\\0\end{bmatrix} + \frac{t^2}{2!}\begin{bmatrix}0\\0\\0\end{bmatrix}\right)$$

$$\vec{x}_1 = e^{2t}\begin{bmatrix}1\\0\\0\end{bmatrix}$$

$$\vec{x}_2 = e^{2t}\left(\begin{bmatrix}0\\1\\0\end{bmatrix} + t\begin{bmatrix}0&1&3\\0&0&-1\\0&0&0\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix} + \frac{t^2}{2!}\begin{bmatrix}0&0&-1\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix}\right)$$

$$= e^{2t}\left(\begin{bmatrix}0\\1\\0\end{bmatrix} + t\begin{bmatrix}1\\0\\0\end{bmatrix} + \frac{t^2}{2!}\begin{bmatrix}0\\0\\0\end{bmatrix}\right)$$

$$= e^{2t}\begin{bmatrix}t\\1\\0\end{bmatrix}$$

$$\vec{x_3} = e^{2t}\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \frac{t^2}{2!} \left(\quad \right) \right.$$

$$\left(\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

$$= e^{2t}\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} + \frac{t^2}{2} \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \right)$$

$$= e^{2t}\left(\begin{bmatrix} 3t - t^2/2 \\ -t \\ 1 \end{bmatrix} \right)$$

Solution: $\vec{x} = c_1 \vec{x_1} + c_2 \vec{x_2} + c_3 \vec{x_3}$

$$\boxed{\vec{x} = c_1 e^{2t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} t \\ 1 \\ 0 \end{bmatrix} + c_3 e^{2t} \begin{bmatrix} 3t - t^2/2 \\ -t \\ 1 \end{bmatrix}}$$

Ex 4: $\vec{x}' = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \vec{x}$

$$|A - rI| = \begin{vmatrix} -r & 0 & 1 \\ 0 & 1-r & 2 \\ 0 & 0 & 1-r \end{vmatrix} =$$

$$(-r)\left((1-r)^2\right) - 0\left(\cdots\right) + 1\left(0-0\right)\Big)$$

I didn't notice A is upper triangular

$$-r(1-r)^2 = 0$$

$r_1 = 0$, $r_2 = r_3 = 1$, mult 2

$$(A - 0I)\vec{u}_1 = \vec{0}$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_a \\ u_b \\ u_c \end{bmatrix} = \vec{0}$$

$1u_b + 2u_c = 0$ $\quad u_b = 0$

$1u_c = 0 \rightarrow$

$u_c = 0$

$$\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$r_2 = r_3 = 1$

$$(A - 1I)^2 \vec{u} = \vec{0}$$

$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}^2 \vec{u} = \vec{0}$$

$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \vec{u} = \vec{0}$$

multiply to get

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_a \\ u_b \\ u_c \end{bmatrix} = \vec{0}$$

$$\vec{u_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$1u_a - 1u_c = 0 \quad u_a = u_c$$

$$\vec{u_3} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

We already know $\vec{x_1} = e^{0t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

We could still plug in w/ $m = 1$

$$\vec{x_1} = e^{0t} \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}} \right) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{x_2} = e^t\left(\begin{bmatrix}0\\1\\0\end{bmatrix} + t\begin{bmatrix}-1 & 0 & 1\\0 & 0 & 2\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix} + \frac{t^2}{2}\begin{bmatrix}1 & 0 & -1\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix}\right)$$

$$= e^t \begin{bmatrix}0\\1\\0\end{bmatrix} + t\begin{bmatrix}0\\0\\0\end{bmatrix} + \frac{t^2}{2}\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$= e^t\begin{bmatrix}0\\1\\0\end{bmatrix}$$

$$\vec{x_3} = e^t\left(\begin{bmatrix}1\\0\\1\end{bmatrix} + t\begin{bmatrix}-1 & 0 & 1\\0 & 0 & 2\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}1\\0\\1\end{bmatrix} + \frac{t^2}{2}\begin{bmatrix}1 & 0 & -1\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}1\\0\\1\end{bmatrix}\right)$$

$$= e^t\left(\begin{bmatrix}1\\0\\1\end{bmatrix} + t\begin{bmatrix}0\\2\\0\end{bmatrix} + \frac{t^2}{2}\begin{bmatrix}0\\0\\0\end{bmatrix}\right)$$

$$= e^t\begin{bmatrix}1\\2t\\1\end{bmatrix}$$

$$\vec{x} = c_1\begin{bmatrix}1\\0\\0\end{bmatrix} + c_2e^t\begin{bmatrix}0\\1\\0\end{bmatrix} + c_3e^t\begin{bmatrix}1\\2t\\1\end{bmatrix}$$