MA 341 Test 2 Version 2

1. (15 points) Use **the method of undetermined coefficients** to solve the Initial Value Problem (IVP): $y'' + 4y' + 4y = 8t$; y(0)=0, $y'(0)$=0

2. (13 points) Use **the method of variation of parameters** to find the particular solution to: $y'' + y = sec(t)$

3. (14 points) A 1 kg mass attached to a spring stretches it 5 m before coming to a rest at equilibrium. The damping constant is 1 N - sec/m. At time t = 0, an external force F(t) = sin(t) N is applied to the system and the mass is pulled 1 m below equilibrium and released. If x(t) is the position of the mass at time t, answer the following:
   a) Use 10 m/s^2 for the gravitational constant and formulate the IVP that describes this system
   b) Determine its steady - state solution

4. (12 points) Use the definition of the Laplace transform of f(t)=u(t-4) and determine its domain

Use the table below to answer the following problems:

$L\{y''\} = s^2L\{y\} - sy(0) - y'(0)$	$L\{\cos(bt)\} = \frac{s}{s^2+b^2}$	$L\{e^{at}\} = \frac{1}{s-a}$
$L\{y'\} = sL\{y\} - y(0)$	$L\{\sin(bt)\} = \frac{b}{s^2+b^2}$	$L\{t^n\} = \frac{n!}{s^{n+1}}$
$L\{t^n e^{at}\} = \frac{n!}{(s-a)^{n+1}}$	$L\{e^{at}\cos(bt)\} = \frac{s-a}{(s-a)^2+b^2}$	$L\{e^{at}\sin(bt)\} = \frac{b}{(s-a)^2+b^2}$
$L\{g(t)u(t-a)\} = e^{-as}L\{g(t+a)\}$	$L^{-1}\{e^{-as}F(s)\} = f(t-a)u(t-a)$	$L\{t^n f(t)\} = (-1)^n \frac{d^n(F(s))}{ds^n}$
	$L\{1\} = \frac{1}{s}$	$L\{u(t-a)\} = \frac{e^{-as}}{s}$

5. (12 points) Express the given function using unit step functions and compute its Laplace transform: $f(t) = \begin{cases} 3t & t < 1 \\ 0 & 1 \le t \end{cases}$

6. (16 points) Find the inverse Laplace of the following: $\frac{8e^{-s}}{s(s-2)^2}$

7. (18 points) Use the method of Laplace transforms to solve the Initial Value Problem: $y'' - 2y' + 5y = 20$; y(0)=0, $y'(0)$=0

341 T2 V2 Solutions

1. (15 points)

$r^2 + 4r + 4 = 0$

$(r+2)^2 = 0$

$y_c = c_1 e^{-2t} + c_2 t e^{-2t}$

$y_p = At + B \quad y_p' = A \quad y_p'' = 0$

$4A + 4At + 4B = 8t$

$A = 2$

$4A + 4B = 0 \quad B = -2$

$y = c_1 e^{-2t} + c_2 t e^{-2t} + 2t - 2$

$y(0) = c_1 - 2 = 0 \quad c_1 = 2$

$y = 2e^{-2t} + c_2 t e^{-2t} + 2t - 2$

$y' = -4e^{-2t} + c_2 e^{-2t} - 2c_2 t e^{-2t} + 2$

$y'(0) = -4 + c_2 + 2 = 0 \quad c_2 = 2$

$$\boxed{y = 2e^{-2t} + 2te^{-2t} + 2t - 2}$$

2. (13 points) $r^2+1=0$

$$y_c = c_1\cos t + c_2\sin t$$

$$y_p = v_1\cos t + v_2\sin t$$

$$v_1' y_1 + v_2' y_2 = 0 \qquad v_1'\cos t + v_2'\sin t = 0$$

$$v_1' = -v_2' \frac{\sin t}{\cos t}$$

$$v_1' y_1' + v_2' y_2' = f/a \qquad v_1'(-\sin t) + v_2'\cos t = \sec t$$

$$\cos t\left(v_2' \frac{\sin^2 t}{\cos t} + v_2'\cos t = \sec t\right)$$

$$v_2'(\sin^2 t + \cos^2 t) = 1$$

$$v_2 = t$$

$$v_1' = -1\,\frac{\sin t}{\cos t}$$

$$v_1 = \int \frac{-\sin t\, dt}{\cos t}$$

$$u = \cos t$$

$$du = -\sin t\, dt$$

$$= \int \frac{1}{u}\, du$$

$$= \ln|\cos t|$$

$$\boxed{y_p = \ln|\cos t|\cos t + t\sin t}$$

3. (14 points) $my'' + by' + ky = F_{ext}$

a) $y'' + y' + 2y = \sin t$

$y(0) = 1$

$y'(0) = 0$

$\overset{m}{1} \cdot \overset{g}{10} = k\,5 \qquad k = 2$

b) $y_p = A\cos t + B\sin t$

$y_p' = -A\sin t + B\cos t$

$y_p'' = -A\cos t - B\sin t$

$-A\cos t - B\sin t - A\sin t + B\cos t +$

$2A\cos t + 2B\sin t = \sin t$

$-A + B + 2A = A + B = 0$

$-B - A + 2B = -A + B = 1$

$2B = 1$

$B = 1/2 \quad A = -1/2$

$$y_p = -\frac{1}{2}\cos t + \frac{1}{2}\sin t$$

Bsint

$y_p' = -A\sin t + B\cos t$

$y_p'' = -A\cos t - B\sin t$

$-A\cos t - B\sin t - A\sin t + B\cos t +$

$2A\cos t + 2B\sin t = \sin t$

$-A + B + 2A = A + B = 0$

$-B - A + 2B = -A + B = 1$

$2B = 1$

$B = 1/2 \quad A = -1/2$

$y_p = -\frac{1}{2}\cos t + \frac{1}{2}\sin t$

4. (12 points)

$$\mathcal{L}\{u(t-4)\} = \int_0^\infty e^{-st} u(t-4)\,dt$$

$$= \int_0^4 e^{-st} \cdot 0\,dt + \int_4^\infty e^{-st}\, 1\,dt$$

$$= \lim_{n\to\infty} \int_4^n e^{-st}\,dt = \lim_{n\to\infty} -\frac{1}{s} e^{-st}\Big|_4^n$$

$$\lim_{n\to\infty} -\frac{1}{s}\left[e^{-sn} - e^{-4s}\right] = \boxed{\frac{1}{s} e^{-4s}} \quad \boxed{s > 0}$$

5. (12 points)

$$f = 3t - 3t\,u(t-1)$$

$$\mathcal{L}\{f\} = \frac{3}{s^2} - e^{-s}\mathcal{L}\{\underbrace{3(t+1)}_{3t+3}\}$$

$$= \frac{3}{s^2} - e^{-s}\left[\frac{3}{s^2} + \frac{3}{s}\right]$$

4. (12 points)

$$\mathcal{L}\{u(t-4)\} = \int_0^\infty e^{-st} u(t-4)\, dt$$

$$= \int_0^4 e^{-st} \cdot 0 \, dt + \int_4^\infty e^{-st} \cdot 1 \, dt$$

$$= \lim_{n\to\infty} \int_4^n e^{-st}\, dt = \lim_{n\to\infty} -\frac{1}{s} e^{-st} \Big|_4^n$$

$$= \lim_{n\to\infty} -\frac{1}{s}\left[e^{-sn} - e^{-4s}\right] = \frac{1}{s} e^{-4s} \quad [s > 0]$$

5. (12 points)

$$f = 3t - 3t\, u(t-1)$$

$$\mathcal{L}\{f\} = \frac{3}{s^2} - e^{-s}\mathcal{L}\{3(t+1)\}$$

$$3t + 3$$

$$= \frac{3}{s^2} - e^{-s}\left[\frac{3}{s^2} + \frac{3}{s}\right]$$

6. (16 points)

$$F(s) = \frac{8}{s(s-2)^2} = \frac{A}{s} + \frac{B}{s-2} + \frac{C}{(s-2)^2}$$

$$A(s^2-4s+4) + Bs(s-2) + Cs = 8$$

$$As^2 - 4As + 4A + Bs^2 - 2Bs + Cs = 8$$

$$A+B=0$$

$$-4A-2B+C=0$$

$$4A=8 \quad A=2 \quad B=-2$$

$$-8+4+C=0 \quad C=4$$

$$F(s) = \frac{2}{s} - \frac{2}{s-2} + \frac{4}{(s-2)^2}$$

$$f(t) = 2 - 2e^{2t} + 4te^{2t}$$

$$\mathcal{L}^{-1} = u(t-1)f(t-1) = u(t-1)\left[2-2e^{2(t-1)} + 4(t-1)e^{2(t-1)}\right]$$

6. (16 points)

$$F(s) = \frac{8}{s(s-2)^2} = \frac{A}{s} + \frac{B}{s-2} + \frac{C}{(s-2)^2}$$

$$A(s^2-4s+4) + Bs(s-2) + Cs = 8$$

$$As^2 - 4As + 4A + Bs^2 - 2Bs + Cs = 8$$

$$A + B = 0$$

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$$4A = 8 \quad A = 2 \quad B = -2$$

$$-8 + 4 + C = 0 \quad C = 4$$

$$F(s) = \frac{2}{s} - \frac{2}{s-2} + \frac{4}{(s-2)^2}$$

$$f(t) = 2 - 2e^{2t} + 4te^{2t}$$

$$y = u(t-1)f(t-1) = u(t-1)\left[2 - 2e^{2(t-1)} + 4(t-1)e^{2(t-1)}\right]$$

7. (18 points)

$$s^2 L - s\cancel{y(0)} - \cancel{y'(0)} - 2[sL - \cancel{y(0)}] + 5L = \frac{20}{s}$$

$$L = \frac{20}{s(s^2-2s+5)} = \frac{A}{s} + \frac{B(s-1)+2C}{(s-1)^2+2^2}$$

$$(s-\alpha)^2 + \beta^2$$

$$s^2 - 2\alpha s + \alpha^2 + \beta^2$$

$$\alpha = 1 \quad \beta = 2$$

$$A(s^2-2s+5) + [B(s-1)+2C]s = 20$$

$$As^2 - 2As + 5A + Bs^2 - Bs + 2Cs = 20$$

$$A + B = 0$$

$$-2A - B + 2C = 0$$

$$A = 4 \quad B = -4$$

$$-8 + 4 + 2C = 0 \qquad C = 2$$

$$\boxed{y = 4 - 4e^t \cos 2t + 2e^t \sin 2t}$$

7. (18 points)

$$s^2L - sy(0) - y'(0) - 2[sL - y(0)] + 5L = \frac{20}{s}$$

$$L = \frac{20}{s(s^2-2s+5)} = \frac{A}{s} + \frac{B(s-1)+2C}{(s-1)^2+2^2}$$

$$(s-\alpha)^2+\beta^2$$

$$s^2-2\alpha s+\alpha^2+\beta^2$$

$$\alpha = 1 \quad \beta = 2$$

$$A(s^2-2s+5) + [B(s-1)+2C]s = 20$$

$$As^2 - 2As + 5A + Bs^2 - Bs + 2Cs = 20$$

$$A+B=0$$

$$-2A - B + 2C = 0$$

$$A=4 \quad B=-4$$

$$-8+4+2C=0 \quad C=2$$

$$y = 4 - 4e^t\cos 2t + 2e^t\sin 2t$$