1. (19 points) Find the general solution to $\vec{x}' = \begin{bmatrix} 0 & -1 & 1 \\ 0 & 2 & 0 \\ -2 & -1 & 3 \end{bmatrix}\vec{x}$ if its characteristic equation is $(r-2)^2(r-1) = 0$

2. (30 points) Use $\vec{x}' = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}\vec{\mathbf{x}} + \begin{bmatrix} 3e^{-2t} \\ 0 \end{bmatrix}$ to answer the following:
   a) Find its complementary solution. Hint: If r=$\alpha + \beta i$ with **u**=**a** + **bi** then two linearly independent solutions are $e^{\alpha t}(cos(\beta t)\mathbf{a} - \sin(\beta t)\mathbf{b})$ and $e^{\alpha t}(cos(\beta t)\mathbf{b} + \sin(\beta t)\mathbf{a})$

   b) Find the particular solution to $\vec{x}' = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}\vec{\mathbf{x}} + \begin{bmatrix} 3e^{-2t} \\ 0 \end{bmatrix}$ using the method of undetermined coefficients. You don't need part a) to do part b).

3. (23 points) Find the particular solution to $\vec{x}' = \begin{bmatrix} 1 & -4 \\ 2 & -5 \end{bmatrix}\vec{x} + \begin{bmatrix} 4e^{-t} \\ e^{-t} \end{bmatrix}$ using

   the method of variation of parameters if $\vec{x}_c = c_1 e^{-t}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{-3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

   Hint: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

4. (16 points) Tank A initially holds 26 L of brine solution containing 1 kg of dissolved salt; tank B initially contains 19 L of pure water. The tanks are connected by pipes. The liquid in tank A flows into tank B at a rate of 3 L/min and from tank B into tank A at a rate of 4 L/min. A solution containing 0.2 kg/L of salt is poured into tank B at a rate of 6 L/min. Both tanks are well -mixed. The contents of tank A flow out of a drain at the bottom of tank A at a rate of 1 L/min while the contents of tank B flow out of a drain at the bottom of tank B at a rate of 5 L/min.

   If $x_1(t)$ is the amount of salt in tank A and $x_2(t)$ is the amount of salt in tank B, find the coefficient matrix **A**, $\vec{\mathbf{f}}$, and $\vec{x}$ (0) so that $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \mathbf{A}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \vec{\mathbf{f}}$ Do not solve this system!

5. (12 points) Find the inverse of A = $\begin{bmatrix} -1 & 1 & 3 \\ 0 & 1 & 0 \\ 2 & -2 & -5 \end{bmatrix}$ using row operations

MA 341 T3 Solutions

1. (19 points)

$$(A-2I)\vec{u}=\vec{0}$$

$$\begin{bmatrix} -2 & -1 & 1 \\ 0 & 0 & 0 \\ -2 & -1 & 1 \end{bmatrix}\begin{bmatrix} u_a \\ u_b \\ u_c \end{bmatrix}=\vec{0}$$

$$-2u_a-1u_b+1u_c=0$$

$$u_c=u_b+2u_a$$

$$\vec{u_1}=\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \qquad \vec{u_2}=\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$(A-I)\vec{u_3}=\vec{0}=\begin{bmatrix} -1 & -1 & 1 \\ 0 & 1 & 0 \\ -2 & -1 & 2 \end{bmatrix}\begin{bmatrix} u_a \\ u_b \\ u_c \end{bmatrix}=\vec{0}$$

$$u_b=0 \qquad -u_a+u_c=0 \qquad u_a=u_c$$

$$\vec{u_3}=\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\vec{x}=c_1e^{2t}\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}+c_2e^{2t}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}+c_3e^{t}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

2. (30 points)

a) $|A - rI| = \begin{vmatrix} -r & 1 \\ -2 & -2-r \end{vmatrix} =$

$(-r)(-2-r) + 2 = 0$

$r^2 + 2r + 2 = 0$

$r = \dfrac{-2 \pm \sqrt{4 - 4 \cdot 2}}{2} = \dfrac{-2 \pm 2i}{2} = -1 \pm i$

$(A - (-1+i)I)\vec{u} = \vec{0}$

$\begin{bmatrix} 1-i & 1 \\ -2 & -1-i \end{bmatrix} \begin{bmatrix} u_a \\ u_b \end{bmatrix} = \vec{0}$

$(-2)u_a + (-1-i)u_b = 0$

$u_a = \frac{-1}{2}(1+i)u_b$

$\vec{u} = \begin{bmatrix} -1-i \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} i$

$\vec{x}_c = c_1 e^{-t}\left(\cos t \begin{bmatrix} -1 \\ 2 \end{bmatrix} - \sin t \begin{bmatrix} -1 \\ 0 \end{bmatrix}\right)$

$+ c_2 e^{-t}\left(\cos t \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \sin t \begin{bmatrix} -1 \\ 2 \end{bmatrix}\right)$

2 b

$$\vec{x_p} = \vec{a} e^{-2t} \qquad \vec{x_p}' = -2\vec{a}e^{-2t}$$

$$-2\vec{a}e^{-2t} = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} \vec{a} e^{-2t} + \begin{bmatrix} 3e^{-2t} \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2a_1 \\ -2a_2 \end{bmatrix} = \begin{bmatrix} a_2 \\ -2a_1 - 2a_2 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 \\ 0 \end{bmatrix} = \begin{bmatrix} +2a_1 + a_2 \\ -2a_1 \end{bmatrix} \qquad \begin{matrix} a_1 = 0 \\ a_2 = -3 \end{matrix}$$

$$\vec{x_p} = \begin{bmatrix} 0 \\ -3 \end{bmatrix} e^{-2t}$$

3. (23 points)

$$X = \begin{bmatrix} 2e^{-t} & e^{-3t} \\ e^{-t} & e^{-3t} \end{bmatrix} \qquad X^{-1} = \frac{1}{2e^{-4t} - e^{-4t}} \begin{bmatrix} e^{-3t} & -e^{-3t} \\ -e^{-t} & 2e^{-t} \end{bmatrix}$$

$$X^{-1} = \begin{bmatrix} e^{t} & -e^{t} \\ -e^{3t} & 2e^{3t} \end{bmatrix}$$

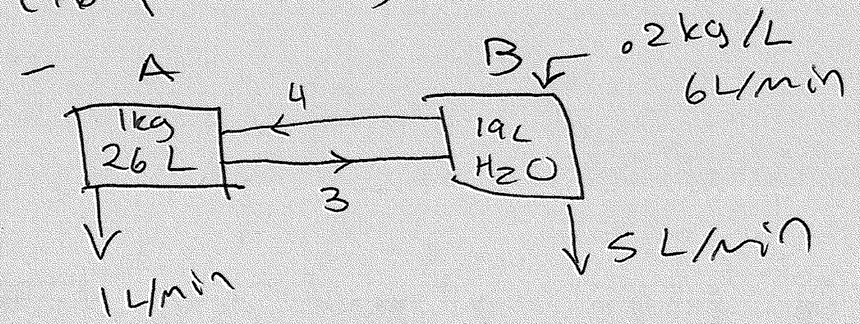
$$\vec{x_p} = X \int X^{-1} \vec{f}\, dt$$

$$= X \int \begin{bmatrix} e^{t} & -e^{t} \\ -e^{3t} & 2e^{3t} \end{bmatrix} \begin{bmatrix} 4e^{-t} \\ e^{-t} \end{bmatrix} dt$$

$$= X \int \begin{bmatrix} 4 - 1 \\ -4e^{2t} + 2e^{2t} \end{bmatrix} dt = X \int \begin{bmatrix} 3 \\ -2e^{2t} \end{bmatrix} dt =$$

$$\begin{bmatrix} 2e^{-t} & e^{-3t} \\ e^{-t} & e^{-3t} \end{bmatrix} \begin{bmatrix} 3t \\ -e^{2t} \end{bmatrix} = \begin{bmatrix} 6te^{-t} - e^{-t} \\ 3te^{-t} - e^{-t} \end{bmatrix}$$

4. (16 points)



$$\frac{dx_1}{dt} = 4\left(\frac{x_2}{19}\right) - 4\left(\frac{x_1}{26}\right)$$

$$\frac{dx_2}{dt} = 6(.2) + 3\left(\frac{x_1}{26}\right) - 9\left(\frac{x_2}{19}\right)$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} \frac{-4}{26} & \frac{4}{19} \\ 3/26 & -9/19 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 6(.2) \end{bmatrix}$$

$$\vec{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

5. (12 points)

$$\left[\begin{array}{ccc|ccc} -1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 2 & -2 & -5 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-R_1} \left[\begin{array}{ccc|ccc} 1 & -1 & -3 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 2 & -2 & -5 & 0 & 0 & 1 \end{array}\right]$$

$$R_3 - 2R_1 \left[\begin{array}{ccc|ccc} +1 & -1 & -3 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 & 1 \end{array}\right] \xrightarrow{R_1 + R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & -3 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 & 1 \end{array}\right]$$

$$R_1 + 3R_3 \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 5 & 1 & 3 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 & 1 \end{array}\right] \quad A^{-1}$$