

MA 242 Test 4 Version 2

1. (14 points) Find the mass of the wire with density $\sigma(x, y, z) = (x + y)z$ in the shape of the line segment from $(3, 0, 0)$ to $(0, 3, 2)$
2. (14 points) Find the work done by $\vec{F}(x, y) = x^3\mathbf{i} + 3y\ln(x)\mathbf{j}$ moving a particle along the curve $x = e^y$ from $(1, 0)$ to $(e^2, 2)$
3. (14 points) Find parametric representations for the following. Be sure to include bounds for t .
 - a) The circle $x^2 + y^2 = 9$
 - b) The curve $y = \sqrt{x+2}$ that lies in the plane $z=1$ from $(2, 2, 1)$ to $(7, 3, 1)$
4. (30 points) Use $\vec{F}(x, y, z) = \left\langle \frac{-1}{y}, \left(\frac{x}{y^2} + \frac{1}{\sqrt{y}} + z \right), (\cos(z) + y) \right\rangle$ to answer the following:
 - a) Show $\vec{F}(x, y, z) = \left\langle \frac{-1}{y}, \left(\frac{x}{y^2} + \frac{1}{\sqrt{y}} + z \right), (\cos(z) + y) \right\rangle$ is a conservative vector field
 - b) Find the most general potential function f of \vec{F}
 - c) Use the Fundamental theorem for Line Integrals to find the work done by $\vec{F}(x, y, z) = \left\langle \frac{-1}{y}, \left(\frac{x}{y^2} + \frac{1}{\sqrt{y}} + z \right), (\cos(z) + y) \right\rangle$ moving along the curve given by
 $\vec{r}(t) = \left\langle t + 1, t + 1, \frac{\pi t}{6} \right\rangle$ where $0 \leq t \leq 3$
5. (14 points) Find the surface area of the portion of the plane $2x + 3y + z = 12$ that lies inside the cylinder $x^2 + y^2 = 4$
6. (14 points) Use Green's theorem to find the circulation of the vector field $\vec{F}(x, y) = (\tan(x) + 2y)\mathbf{i} + (2x + x^2)\mathbf{j}$ moving along the curve $y = 4 - x^2$ from $(0, 4)$ to $(2, 0)$ and then along the line segments from $(2, 0)$ to $(0, 0)$ and then from $(0, 0)$ to $(0, 4)$

Hint: $\oint_C \vec{F} \cdot d\vec{r} = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dA$

C3 T4 Solutions

1. (14 points)

$$\begin{aligned}\vec{r} &= (1-t)\vec{r}_0 + \vec{r}_1 t \\ &= (1-t)\langle 3, 0, 0 \rangle + \langle 0, 3, 2 \rangle + \\ &= \langle 3 - 3t, 0, 0 \rangle + \langle 0, 3t, 2t \rangle \\ &= \langle 3 - 3t, 3t, 2t \rangle\end{aligned}$$

$$\begin{aligned}m &= \int_C (x+y) z \, ds = \int_0^1 6 + \sqrt{(-3)^2 + 3^2 + 2^2} \, dt \\ &= 3t^2 \Big|_0^1 \sqrt{22} = \boxed{3\sqrt{22}}\end{aligned}$$

2. (14 points)

$$\begin{aligned}W &= \int_C \vec{F} \cdot d\vec{r} = \int_0^2 \langle e^{3t}, 3t \cdot t \rangle \cdot \langle e^t, 1 \rangle \, dt \\ x &= e^t \\ y &= t \\ &= \int_0^2 e^{4t} + 3t^2 \, dt \\ &= \frac{1}{4}e^{4t} + t^3 \Big|_0^2 \\ &= \boxed{\frac{1}{4}e^8 + 8 - \frac{1}{4}}\end{aligned}$$

3. (14 points)

a) $x = 3\cos t, y = 3\sin t, 0 \leq t \leq 2\pi$

b) $x = t, y = \sqrt{t+2}, z = 1, 2 \leq t \leq 7$

4. (30 points)

a) $\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{vmatrix}$

b) $\langle 1-1, -(0-0), \frac{1}{y^2} - (\frac{1}{y^2}) \rangle = \vec{0} \quad \checkmark$

$f_x = -\frac{1}{y^2} \quad f_y = \frac{x}{y^2} + \frac{1}{y^2} + z \quad f_z = \cos z + y$

$$f = -\frac{x}{y^2} + g(y, z)$$

$$f_y = \frac{x}{y^2} + g_y = \frac{x}{y^2} + \frac{1}{y^2} + z$$

$$g = 2\sqrt{y} + yz + h(z)$$

$$f = -\frac{x}{y^2} + 2\sqrt{y} + yz + h(z)$$

$$f_z = y + h'(z) = \cos z + y$$

$$h(z) = \sin z + k$$

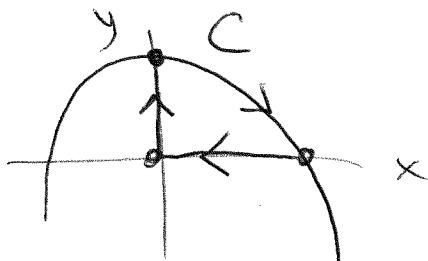
$$f = -\frac{x}{y^2} + 2\sqrt{y} + yz + \sin z + k$$

c) $\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a)) = f(\vec{r}(3)) - f(\vec{r}(0)) =$
 $f(4, 4, \pi/2) - f(1, 1, 0) = \frac{-4}{4} + 2\sqrt{4} + 4(\frac{\pi}{2}) + \sin \frac{\pi}{2} - (-\frac{1}{1} + 2\sqrt{1} + 0 + 0)$
 $= 3 + 2\pi$

5. (14 points) $z = 12 - 2x - 3y$

$$\begin{aligned} S.A. &= \iint_D \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1} dA \\ &= \iint_D \sqrt{(-2)^2 + (-3)^2 + 1} dA \\ &= \iint_D \sqrt{14} dA \\ &= \sqrt{14} \pi \cdot 2^2 = \boxed{4\pi\sqrt{14}} \\ \text{or } & \int_0^{2\pi} \int_0^2 \sqrt{14} r dr d\theta = \dots \end{aligned}$$

6. (14 points)



$$\oint_C \vec{F} \cdot d\vec{r} = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dA = \iint_D 2 + 2x - 2 dA$$

$$-\int_0^2 \int_0^{\sqrt{4-x^2}} 2x dy dx = -\int_0^2 2xy \Big|_0^{\sqrt{4-x^2}} =$$

$$-\int_0^2 2x(4-x^2) dx = -\int_0^2 8x - 2x^3 dx$$

$$= -4x^2 + \frac{2}{4}x^4 \Big|_0^2 = -16 + 8 = \boxed{-8}$$