MA 341 Test 1 Version 1

1. (23 points)
   a) Solve the Initial Value Problem (IVP): $\frac{dy}{dx} = \frac{x}{2y\sqrt{x^2+1}}$, $y(0) = -2$
      Write your answer with y as an explicit function of x if possible.
   b) Does the Existence and Uniqueness Theorem guarantee that this is a unique solution? Justify your answer.

2. (22 points) A large mixing tank initially holds 140 liters of water. At t=0 a brine solution is pumped into the tank at a rate of 30 L/min. The solution in the tank is kept well-mixed and flows out of the tank at a rate of 20 L/min. If the concentration of salt entering the tank is 0.1 kg/L and x(t) represents the amount of salt in the tank at time t, find x(t).

3. (14 points) Use $\left(ye^{xy} + \frac{1}{y}\right) dx + \left(xe^{xy} - \frac{x}{y^2} + \cos(y)\right) dy = 0$ to answer the following:
   a) What is the test for exactness in general?
   b) Assume this is an exact differential equation and find its implicit solution.

4. (14 points) Solve the IVP: $y'' - 6y' + 25y = 0,\ y(0) = 3, y'(0) = 1$

5. (13 points) Use the differential equation $\frac{dy}{dt} = (y-2)^2(y+4)$ to answer the following:
   a) Sketch its phase line and classify its equilibria as we have done in class
   b) Use the phase line to determine the asymptotic behavior as $t \to \infty$ of the solution through y(0)=0
   c) If y(0)=2, what is y(27)?

6. (14 points) Determine if y=sin(xy) is a solution to the IVP:
   $\frac{dy}{dx} = \frac{y}{\sec(xy)-x}$; $y\left(\frac{\pi}{2}\right)=1$

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1\. (23 points)

a)

$$\int 2y\,dy = \int \frac{x}{\sqrt{x^2+1}}\,dx \qquad u = x^2+1$$

$$du = 2x\,dx$$

$$\tfrac{1}{2}du = x\,dx$$

$$y^2 = \int \tfrac{1}{2} u^{-1/2}\,du$$

$$= u^{1/2} + C$$

$$= \sqrt{x^2+1} + C$$

$$y = \pm\sqrt{\sqrt{x^2+1} + C}$$

$$y(0) = -2 = -\sqrt{\sqrt{1} + C}$$

$$C = 3$$

$$\boxed{y = -\sqrt{\sqrt{x^2+1} + 3}}$$

b)

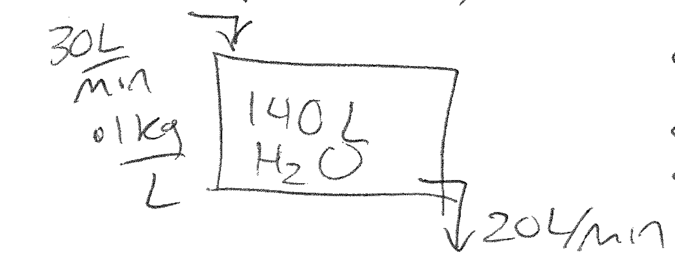
$$f = \frac{x}{2y\sqrt{x^2+1}} \qquad f \text{ is cont at \& around } (0,-2)$$

$$= \frac{xy^{-1}}{2\sqrt{x^2+1}}$$

$$\frac{\partial f}{\partial y} = \frac{-xy^{-2}}{2\sqrt{x^2+1}} \qquad \frac{\partial f}{\partial y} \text{ is cont at \& around } (0,-2)$$

Yes.

2. (22 points)



$$\frac{dx}{dt} = F_iC_i - F_oC_o$$

$$\frac{dx}{dt} = 30(.1) - 20\left(\frac{x}{140+10t}\right)$$

$$= 3 - \frac{2x}{14+t}$$

$$x(0) = 0$$

$$\frac{dx}{dt} + \frac{2x}{14+t} = 3$$

$$\mu(t) = e^{\int \frac{2}{14+t}} = e^{2\ln(14+t)} = (14+t)^2$$

$$\frac{d}{dt}\left[\mu(t)x\right]$$

$$\frac{d}{dt}\left[(14+t)^2 x\right] = 3(14+t)^2$$

$$(14+t)^2 x = (14+t)^3 + C$$

$$x = \frac{(14+t)^3 + C}{(14+t)^2}$$

$$x(0) = 0 = \frac{14^3 + C}{14^2}$$

$$\boxed{x = \frac{(14+t)^3 - 14^3}{(14+t)^2}}$$

3. (14 points)

a) $M_y = N_x$

b) $F_x = ye^{xy} + \frac{1}{y}$ $\quad F_y = xe^{xy} - \frac{x}{y^2} + \cos y$

$F = e^{xy} + \frac{x}{y} + g(y)$

$F_y = xe^{xy} - xy^{-2} + g'(y) =$

$g'(y) = \cos y$

$g(y) = \sin y$

$$\boxed{e^{xy} + \frac{x}{y} + \sin y = C}$$

4. (14 points)

$r^2 - 6r + 25 = 0$

$r = \frac{6 \pm \sqrt{36 - 100}}{2} = \frac{6 \pm 8i}{2} = 3 \pm 4i$

$y = e^{3t}(c_1 \cos 4t + c_2 \sin 4t)$

$y(0) = 3 = c_1$

$y' = 3e^{3t}(c_1 \cos 4t + c_2 \sin 4t) + e^{3t}(-4c_1 \sin 4t + 4c_2 \cos 4t)$

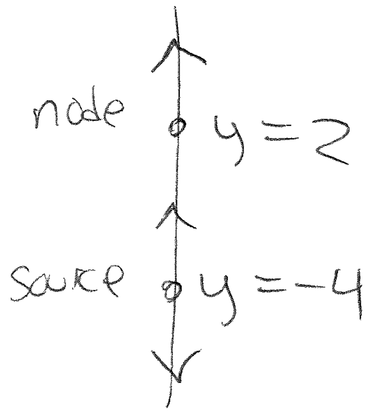
$y'(0) = 1 = 3c_1 + 4c_2$

$1 = 9 + 4c_2 \quad c_2 = -2$

$$\boxed{y = e^{3t}(3\cos 4t - 2\sin 4t)}$$

5. (13 points)

a)



$y > 2$

b) $y \to 2$

c) $y(27) = 2$

6. (14 points)

$y = \sin xy$

$\frac{dy}{dx} = \cos(xy)\left(y + x\frac{dy}{dx}\right)$

$= y\cos xy + x\cos xy \frac{dy}{dx}$

$\frac{dy}{dx} - x\cos xy \frac{dy}{dx} = y\cos xy$

$\frac{dy}{dx} = \frac{y\cos(xy)}{1 - x\cos(xy)} \cdot \frac{\sec xy}{\sec xy} = \frac{y}{\sec(xy) - x}$ ✓

$y\left(\frac{\pi}{2}\right) = 1$: $1 \stackrel{?}{=} \sin\left(\frac{\pi}{2} \cdot 1\right) = 1$ ✓

Yes