MA 341 Test 1 Version 2

1. (23 points)
   a) Solve the Initial Value Problem (IVP): $\frac{dy}{dx} = \frac{x}{2y\sqrt{x^2+1}}$, $y(0) = -3$
      Write your answer with y as an explicit function of x if possible.
   b) Does the Existence and Uniqueness Theorem guarantee that this is a unique solution? Justify your answer.

2. (22 points) A large mixing tank initially holds 120 liters of water. At t=0 a brine solution is pumped into the tank at a rate of 40 L/min. The solution in the tank is kept well-mixed and flows out of the tank at a rate of 30 L/min. If the concentration of salt entering the tank is 0.1 kg/L and x(t) represents the amount of salt in the tank at time t, find x(t).

3. (14 points) Use $\left(ye^{xy} + \frac{1}{y}\right)dx + \left(xe^{xy} - \frac{x}{y^2} + \cos(y)\right)dy = 0$ to answer the following:
   a) What is the test for exactness in general?
   b) Assume this is an exact differential equation and find its implicit solution.

4. (14 points) Solve the IVP: $y'' + 2y' + 10y = 0,\ y(0) = 4, y'(0) = 2$

5. (13 points) Use the differential equation $\frac{dy}{dt} = (y-3)^2(y+5)$ to answer the following:
   a) Sketch its phase line and classify its equilibria as we have done in class
   b) Use the phase line to determine the asymptotic behavior as $t \to \infty$ of the solution through y(0)=0
   c) If y(0)=3, what is y(16)?

6. (14 points) Determine if y=sin(xy) is a solution to the IVP:
   $\frac{dy}{dx} = \frac{y}{\sec(xy)-x}$; $y\left(\frac{\pi}{2}\right)=1$

MA 341 T1 V2 Solutions

1. (23 points)

a) $\int 2y\,dy = \int \frac{x}{\sqrt{x^2+1}}\,dx$ $\quad u = x^2+1$, $du = 2x\,dx$, $\frac{1}{2}du = x\,dx$

$y^2 = \int \frac{1}{2} u^{-1/2}\,du$

$y^2 = u^{1/2} + C$

$y^2 = \sqrt{x^2+1} + C$

$y = \pm\sqrt{\sqrt{x^2+1} + C}$

$y(0) = -3 = -\sqrt{\sqrt{1} + C}$

$C = 8$

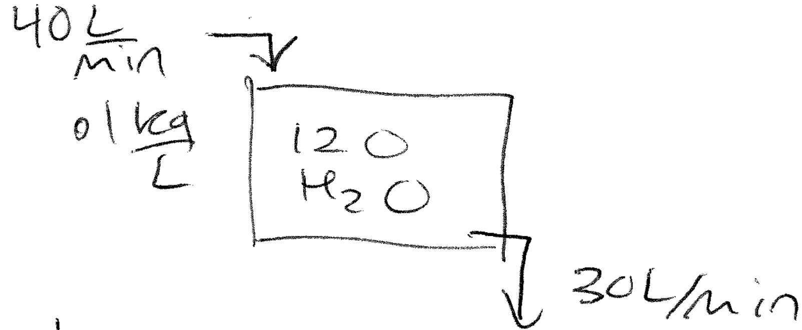
$$\boxed{y = -\sqrt{\sqrt{x^2+1} + 8}}$$

b) $f = \frac{x}{2y\sqrt{x^2+1}}$ $\quad f$ is cont at & around $(0,-3)$

$\frac{\partial f}{\partial y} = \frac{-x y^{-2}}{2\sqrt{x^2+1}}$ $\quad \frac{\partial f}{\partial y}$ is cont at & around $(0,-3)$

Yes.

2. (22 points)



$$\frac{dx}{dt} = F_iC_i - F_oC_o$$

$$\frac{dx}{dt} = 40(.1) - 30\left(\frac{x}{120+10t}\right) \qquad x(0) = 0$$

$$\frac{dx}{dt} = 4 - \frac{3x}{12+t}$$

$$\frac{dx}{dt} + \frac{3x}{12+t} = 4 \qquad \mu(t) = e^{\int \frac{3}{12+t}}$$

$$= e^{3\ln(12+t)}$$

$$= (12+t)^3$$

$$\frac{d}{dt}\left[(12+t)^3 x\right] = 4(12+t)^3$$

$$(12+t)^3 x = (12+t)^4 + C$$

$$x = \frac{(12+t)^4 + C}{(12+t)^3}$$

$$x(0)=0 \rightarrow \frac{12^4 + C}{12^3} = 0 \qquad C = -12^4$$

$$\boxed{x = \frac{(12+t)^4 - 12^4}{(12+t)^3}}$$

3. (14 points)

a) $M_y = N_x$

b) $F_x = ye^{xy} + \frac{1}{y}$

$F = e^{xy} + \frac{x}{y} + g(y))$

$F_y = xe^{xy} - xy^{-2} + g'(y) = xe^{xy} - \frac{x}{y^2} + \cos y$

$g'(y) = \cos y$

$g(y) = \sin y$

$$\boxed{e^{xy} + \frac{x}{y} + \sin y = C}$$

4. (14 points) $r^2 + 2r + 10 = 0$

$r = \frac{-2 \pm \sqrt{4-40}}{2} = \frac{-2 \pm 6i}{2}$

$= -1 \pm 3i$

$y = e^{-t}[C_1 \cos 3t + C_2 \sin 3t]$

$y(0) = 4 = C_1$

$y' = -e^{-t}[\cancel{C_1}^{4} \cos 3t + C_2 \sin 3t] + e^{-t}[-3\cancel{C_1}^{4} \sin 3t + 3C_2 \cos 3t]$

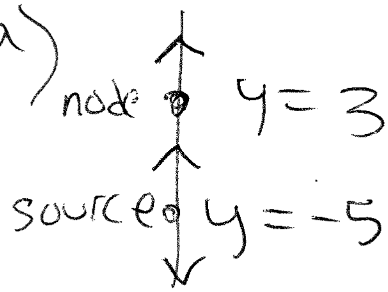
$y'(0) = 2 = -4 + 3C_2$

$6 = 3C_2 \quad C_2 = 2$

$$\boxed{y = e^{-t}[4\cos 3t + 2\sin 3t]}$$

5. (13 points)

a)



b) $y \to 3$

c) $y(16)=3$

6. (14 points)

$$\frac{dy}{dx} = \cos(xy)\left(y + x\frac{dy}{dx}\right)$$

$$\frac{dy}{dx} = \cos(xy)y + x\cos xy \frac{dy}{dx}$$

$$\frac{dy}{dx} - x\cos xy \frac{dy}{dx} = \cos(xy) y$$

$$\frac{dy}{dx} = \frac{\cos(xy) y}{1 - x\cos xy} \cdot \frac{\sec(xy)}{\sec(xy)}$$

$$= \frac{y}{\sec(xy) - x} \checkmark$$

$y = \sin xy$

$1 \stackrel{?}{=} \sin\left(\frac{\pi}{2} \cdot 1\right) = 1 \checkmark$

Yes