MA 242-050 Test 4 Version 1

1. (24 points) Use $\vec{F}(x,y,z)=\langle \tan^{-1} y, \frac{x}{1+y^2}+e^{4z}+4, 4ye^{4z}+4z\rangle$ to answer the following:
   a) Assume $\vec{F}(x,y,z)$ is conservative and find its most general potential function f
   b) Use the Fundamental theorem for Line Integrals to find the work done by $\vec{F}(x,y,z)$ moving along the curve given by $\vec{r}(t)=\langle t, 2-t, \sqrt{2t}\rangle$ where $0\le t\le 2$

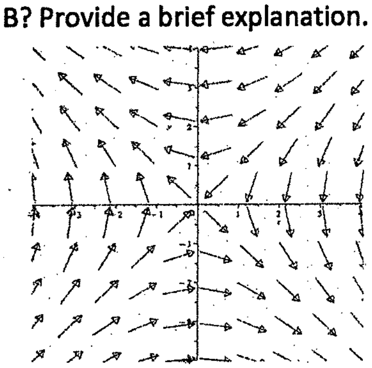
2. (20 points) Use $\vec{F}(x,y,z)=\langle x^2, xy, z-y\rangle$ to answer the following:
   a) Show $\vec{F}(x,y,z)=\langle x^2, xy, z-y\rangle$ is not path independent
   b) Find the work done by $\vec{F}(x,y,z)$ moving a particle along the line segment from (1,0,1) to (1,3,5)

3. (14 points) Find the mass of the wire with density $\sigma(x,y,z)=\frac{yz}{\sqrt{x^2+1}}$ in the shape of the curve $x=e^y$ in the plane z=3 from (1,0,3) to $(e^2,2,3)$

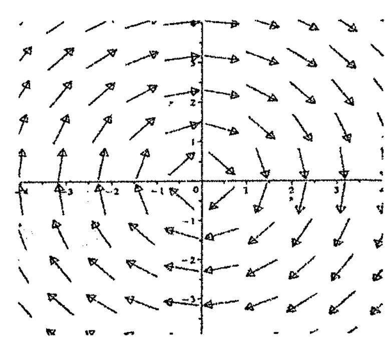
4. (13 points) Use Green's theorem to find the circulation of the vector field $\vec{F}(x,y)=(\sqrt{x}+2y)\mathbf{i}+(5x+\sec^2 y)\mathbf{j}$ moving clockwise along the triangle with vertices (0,0), (0,4), and (5,0). Hint: $\oint_C \vec{F}\bullet d\vec{r}=\iint_D \frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\, dA$

5. (17 points) Find the surface area of the cap of the sphere $x^2+y^2+z^2=9$ where $2\le z\le 3$

6. (12 points) Use the vector fields below to answer the following:
   a) Find the gradient vector field of f(x,y)= -xy
   b) Match your answer in part a) to one of the vector fields below. Provide a brief explanation for your choice
   c) Would $\oint_C \vec{F}\bullet d\vec{r}$ be positive, negative, or zero if C was the counter-clockwise oriented circle centered at the origin with radius 2 if $\vec{F}$ was the vector field shown in choice B? Provide a brief explanation.



A



B

C3 T4 V1 Sp 24

1. (24 points)

a) $f_x = \tan^{-1} y \quad f_y = \frac{x}{1+y^2} + e^{4z} + 4 \quad f_z = 4ye^{4z} + 4z$

$f = x\tan^{-1} y + g(y,z)$

$f_y = \frac{x}{1+y^2} + g_y(y,z) = \frac{x}{1+y^2} + e^{4z} + 4$

$g_y = e^{4z} + 4$

$g = ye^{4z} + 4y + h(z)$

$f = x\tan^{-1} y + ye^{4z} + 4y + h(z)$

$f_z = 0 + 4ye^{4z} + 0 + h'(z) =$

$h'(z) = 4z$

$h(z) = 2z^2 + K$

$\boxed{f = x\tan^{-1} y + ye^{4z} + 4y + 2z^2 + K}$

b) $\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$

$\vec{r}(2) = \langle 2, 0, 2 \rangle$

$\vec{r}(0) = \langle 0, 2, 0 \rangle$

$f(2,0,2) - f(0,2,0)$

$[2\tan^{-1} 0 + 0e^8 + 4 \cdot 0 + 2 \cdot 2^2] - [0 + 2e^0 + 8 + 0]$

$= \boxed{2}$

2. (20 points)

a) $\text{curl}\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & xy & z-y \end{vmatrix}$

$= \langle -1, -(0-0), y \rangle = \langle -1, 0, y \rangle \neq \vec{0}$

b) $\vec{r}(t) = (1-t)\langle 1,0,1\rangle + \langle 1,3,5\rangle t$

$= \langle 1-t, 0, 1-t\rangle + \langle t, 3t, 5t\rangle$

$= \langle 1, 3t\ 1+4t \rangle$

$W = \int_C \vec{F} \circ d\vec{r} = \int_C \langle x^2, xy, z-y\rangle \circ d\vec{r}$

$= \int_0^1 \langle 1, 3t, \underbrace{1+4t-3t}_{1+t} \rangle \circ \langle 0, 3, 4\rangle\, dt$

$= \int_0^1 9t + 4 + 4t$

$= \int_0^1 13t + 4\, dt = \frac{13}{2}t^2 + 4t \Big|_0^1$

$= \boxed{\frac{13}{2} + 4}$

3. (14 point)

$\int_C \sigma\, ds$

$x = e^t$
$y = t$
$z = 3$

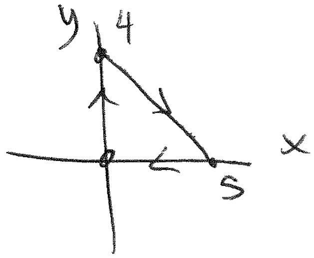
$\vec{r}(t) = \langle e^t, t, 3\rangle$

$\vec{r}'(t) = \langle e^t, 1, 0\rangle$

$\int_0^2 \frac{t \cdot 3}{\cancel{\sqrt{e^{2t}+1}}} \cancel{\sqrt{(e^t)^2 + 1^2 + 0^2}}\, dt$

$= \frac{3}{2}t^2 \Big|_0^2 = \boxed{6}$

4. (13 points)



$-\iint 5-2\, dA$

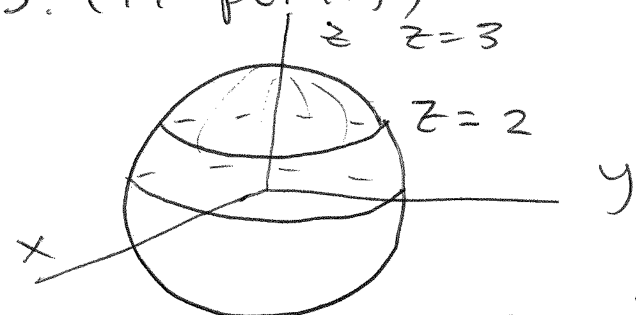
$= -\iint_D 3\, dA$

$m = \frac{\Delta y}{\Delta x} = \frac{0-4}{5-0} = -\frac{4}{5}$

$= -3 \cdot \frac{1}{2} \cdot 4 \cdot 5 = \boxed{-30}$

or $-\int_0^5 \int_0^{4-\frac{4}{5}x} 3\, dy\, dx$

5. (17 points)



$2 = \sqrt{9-r^2}$

$4 = 9-r^2$ $\quad r = \sqrt{5}$

$z = \sqrt{9-x^2-y^2}$

$\frac{\partial z}{\partial x} = \frac{-x}{\sqrt{9-x^2-y^2}}$

$S.A. = \iint_D \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dA$

$= \iint_D \sqrt{\frac{x^2}{9-x^2-y^2} + \frac{y^2}{9-x^2-y^2} + 1}\, dA$

$= \iint_D \sqrt{\frac{x^2+y^2}{9-x^2-y^2} + \frac{9-x^2-y^2}{9-x^2-y^2}}\, dA$

$= \iint \frac{3}{\sqrt{9-x^2-y^2}}\, dA = \int_0^{2\pi} \int_0^{\sqrt{5}} \frac{3}{\sqrt{9-r^2}}\, r\, dr\, d\theta$

$2\pi \left(-\frac{1}{2}\right) \int_9^4 3u^{-1/2}\, du$

$u = 9-r^2$

$du = -2r\, dr$

$\pi \int_4^9 3u^{-1/2}\, du = 6\pi \sqrt{u}\Big|_4^9 = 6\pi\left[\sqrt{9}-\sqrt{4}\right] = \boxed{6\pi}$

$-\frac{1}{2} du = r\, dr$

6. (12 points)

a) $\nabla f = \langle f_x, f_y \rangle$

$= \langle -y, -x \rangle$

b) $-y > 0$ $\quad y < 0$ arrows point right

$\boxed{A}$

c) negative, moving in the opposite direction as $\vec{F}$